Module 6.3: Contrapositives, Converses, and Counter-Examples

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1 What is a Contrapositive? A Counter-Example? A Converse?

"if P then Q" is logically equivalent to "if not Q then not P"

Our goal is to get to the point where we can do the contrapositive mentally. In other words, we want to be able to read a conditional statement (if P then Q) and immediately "see" the contrapositive. There are many reasons that this is useful.

The primary reason is that an entire method of proof, called proof by contraposition, is based upon exactly this. Moreover, I have found that these logical exercises make me a much more effective debugger of code that involves conditional statements. After all, nearly every computer program contains an if-then-else construct. Also, it is very easy to write examination questions involving a contrapositive.

2 Easy Exercises

Please write out the contrapositive of these if-then statements.

1. If a student is in the GDD program at Stout, then that student is in Stout's College of S.T.E.M.&M.
2. If $f(x)$ is differentiable, then $f(x)$ is continuous.
3. (Let x be an integer.) If x is even, then x^2 is divisible by 4.

π. Hypothesis: If a child is good, then that child will receive lots of presents in late December.

4. If x is an integer, then $x^3 - x$ is divisible by 3.
5. If a Stout student's cumulative GPA is below 2.0, then that student is on academic probation.
6. (Let $f(x) = (x-4)^2 - 1$.) If $x > 5$, then $f(x) > 0$.

Answers are provided at the end of this workbook.

3 Recap of DeMorgan's Laws

The above exercises weren't too hard. The more interesting examples of contrapositives occur when we add in DeMorgan's Laws:

- $(\mathcal{A} \cap \mathcal{B})^c = \mathcal{A}^c \cup \mathcal{B}^c$
- $(\mathcal{A} \cup \mathcal{B})^c = \mathcal{A}^c \cap \mathcal{B}^c$
- $\sim (P \vee Q) = (\sim P) \wedge (\sim Q)$
- $\sim (P \wedge Q) = (\sim P) \vee (\sim Q)$

4 Harder Exercises—Contrapositives with DeMorgan's Laws

Using DeMorgan's Laws, please write the contrapositives of these if-then statements in two different ways. (Once without applying DeMorgan's Laws, and once after using DeMorgan's Laws to simplify.)

7. (Let x and y be integers.) If x is even and y is even, then xy is even.
8. If $f(x)$ has a local optimum at $x = 5$, then either $f'(5) = 0$ or $f'(5)$ does not exist.
9. If the cumulative GPA is below 2.0 for the last two semesters, or if the previous semester GPA is below 1.0, then the student gets dismissed.
10. If $xy = 0$, then either $x = 0$ or $y = 0$.
11. If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$.
12. (Let $f(x) = (x - 4)^2 - 1$.) If $x > 3$ and $x < 5$, then $f(x) < 0$.
13. If x^2 is a negative real number, then x is a purely imaginary number.

Answers are provided at the end of this workbook.

5 Counter-Examples

A counter-example for the statement "if P then Q" must be something that makes P true and Q false. It is worth taking a moment to define counter-examples for exercises 1–4.

1. A student who is in the GDD program at Stout and not in Stout's College of S.T.E.M.&M.
2. Some $f(x)$ that is differentiable, but not continuous.
3. Some integer x that is even, yet x^2 is not divisible by 4.

π. A child that is good but who did not receive lots of presents in late December.

4. Some integer x where $x^3 - x$ is not divisible by 3.

For exercises 1, 2, 3, and 4, these statements are true, so you will find no counter-examples no matter how long you search for one.

However, I am sure that in poor neighborhoods we can find counter-examples to the hypothesis marked π, namely children that are good but that did not receive lots of presents in late December. Therefore we must reject the hypothesis marked π as a false statement.

The purpose of including π in this workbook is to show that sometimes a statement is *obviously* false when written as a contrapositive, but perhaps it is *not so obviously* false in the original.

6 Practice with Counter-Examples

Please define counter-examples for exercises 5–13. Because the statements 5–13 were true, a counter-example cannot actually be found. However, we can still write down what a counter-example must look like. Answers are provided at the end of the workbook.

7 Going Deeper: the Inverse and the Converse

For any logical statement, we can actually write it four different ways:

- The original: if P then Q.
- The contrapositive: if not Q then not P.
- The inverse: if not P then not Q.
- The converse: if Q then P.

It turns out that the "original" and the "contrapositive" always have the same truth value as each other. Also, the "inverse" and the "converse" always have the same truth value as each other.

In other words, when the original is true, we know that the contrapositive will be true. However, we have no right to expect either the inverse or the converse to be true. Nonetheless, if we discover either the inverse or the converse to be true, then for sure, both are true.

The converse is used primarily in three places. First, when a mathematician proves a major theorem, often the next step is to explore if the converse is true, as well as why or why not. Second, a major error in proof writing is when a student proves "if Q then P" when the instructor asks for a proof of "if P then Q." Third, there is a method of proof writing, called a bi-conditional proof, which requires the converse. The inverse almost never comes up in mathematics—except in a lesson like this one in a textbook. That's why I didn't include "inverses" in the title of this module.

To explore this concept more fully, it is helpful (though tiresome) to write all 14 statements in all 4 ways. While somewhat tedious, it is a healthy exercise to go through all of them. However, in our case, we've already done all the contrapositives. Therefore, take a blank piece of paper or two, and just write out only the inverse and the converse for statements 1, 2, 3, π, 4, 5, ..., 12, 13. By the way, something cool happens when investigating # 4, and I describe that in the solutions. Of course, since we were able to write the contrapositive two different ways (using DeMorgans Law) for 5–13, then the inverse and the converse can be written two different ways each.

By the way, a cute but mostly useless fact: the contrapositive of the inverse is the converse, and the contrapositive of the converse is the inverse. Try not to think about this particular point too much. It might make your head explode.

8 A Common but Dangerous Abbreviation

If you know that you have a proof that "if P then Q" as well as a proof that "if Q then P," in this uncommon case, you can write an abbreviation. You can say "P if and only if Q." However, it is very important to note that this is an uncommon situation. In general, there are many situations where "if P then Q" is true, but "P if and only if Q" is false. You'll see that for yourself as you do the exercises.

By the way, if you are exceptionally lazy, you can also write "P iff Q" but this is often autocorrected by spelling checkers (such as in gmail), and it alienates persons who are of intermediate experience in mathematics.

9 Answers to Contrapositives

1. If a student is not in Stout's College of S.T.E.M.&M., then that student is not in the Stout GDD program.
2. If $f(x)$ is not continuous, then $f(x)$ is not differentiable.
3. (Let x be an integer.) If x^2 is not divisible by four, then x is odd.

π. Hypothesis: If a child does not receive lots of presents in late December, then the child is not good.

4. If $x^3 - x$ is not divisible by 3, then x is not an integer.

5. If a Stout student is not on academic probation, then that student's cumulative GPA is 2.0 or higher.

6. (Let $f(x) = (x-4)^2 - 1$.) If $f(x) \leq 0$, then $x \leq 5$.

7a. (Let x and y be integers.) If xy is odd, then it is not the case that both x is even and y is even.

7b. (Let x and y be integers.) If xy is odd, then x is odd or y is odd.

8a. If it is not the case that either $f'(5) = 0$ or $f'(5)$ does not exist, then $f(x)$ does not have a local optimum at $x = 5$.

8b. If $f'(5)$ exists and $f'(5) \neq 0$, then $f(x)$ does not have a local optimum at $x = 5$.

9a. If the student does not get dismissed, then it is not the case that either the cumulative GPA is below 2.0 for the last two semesters, or the previous semester GPA is below 1.0.

9b. If the student does not get dismissed, then the cumulative GPA is 2.0 or better for the last two semesters, and the previous semester GPA is 1.0 or above.

10a. If it is not the case that either $x = 0$ or $y = 0$, then $xy \neq 0$.

10b. If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.

11a. If $xy < 0$, then it is not the case that both $x \geq 0$ and $y \geq 0$.

11b. If $xy < 0$, then either $x < 0$ or $y < 0$.

12a. (Let $f(x) = (x-4)^2 - 1$.) If $f(x) \geq 0$, then it is not the case that both $x > 3$ and $x < 5$.

12b. (Let $f(x) = (x-4)^2 - 1$.) If $f(x) \geq 0$, then either $x \leq 3$ or $x \geq 5$.

13a. If x is not a purely imaginary number, then it is not the case that both x^2 is real and x^2 is negative.

13b. If x is not a purely imaginary number, then either x^2 is not real or $x^2 \geq 0$.

10 Answers to Counter-Examples

Note: Statements 1–4 have their counter-examples given on Page 2.

5. A Stout student that has a cumulative GPA below 2.0, and who is not on academic probation.

6. (Let $f(x) = (x-4)^2 - 1$.) An x where $x > 5$ and $f(x) \leq 0$.

7. Integers x and y with x even, y even, and xy odd.

8. A function $f(x)$ with a local optimum at $x = 5$, $f'(5)$ exists, and $f'(5) \neq 0$.

9. A student with cumulative GPA below 2.0 for the last two semesters, or with the previous semester GPA below 1.0, and the student does not get dismissed.

10. An x and a y with $xy = 0$, $x \neq 0$, and $y \neq 0$.

11. An $x \geq 0$, and a $y \geq 0$, with $xy < 0$.

12. (Let $f(x) = (x-4)^2 - 1$.) An x such that $x > 3$, $x < 5$, and $f(x) \geq 0$.

13. Some complex number x where x^2 is a negative real number, but x is not a purely imaginary number.

11 Answers to Writing Out All Four Versions

- Question 1:
  - Original: If a student is in the GDD program at Stout, then that student is in Stout's College of S.T.E.M.&M.
  - Contrapositive: If a student is not in Stout's College of S.T.E.M.&M., then that student is not in the Stout GDD program.
  - Inverse: If a student is not in the Stout GDD program, then that student is not in Stout's College of S.T.E.M.&M.
  - Converse: If a student is in Stout's College of S.T.E.M.&M., then that student is in the Stout GDD program.
  - Note the classic structure: the original and the contrapositive are true, while the inverse and the converse are false.
- Question 2:
  - Original: If $f(x)$ is differentiable, then $f(x)$ is continuous.
  - Contrapositive: If $f(x)$ is not continuous, then $f(x)$ is not differentiable.
  - Inverse: If $f(x)$ is not differentiable, then $f(x)$ is not continuous.
  - Converse: If $f(x)$ is continuous, then $f(x)$ is differentiable.
  - Note the classic structure: the original and the contrapositive are true, while the inverse and the converse are false. To see this, consider $f(x) = |x|$, which is obviously continuous but which has no derivative at $x = 0$.
- Question 3:
  - Original: (Let x be an integer.) If x is even, then x^2 is divisible by 4.
  - Contrapositive: (Let x be an integer.) If x^2 is not divisible by 4, then x is odd.
  - Inverse: (Let x be an integer.) If x is odd, then x^2 is not divisible by 4.
  - Converse: (Let x be an integer.) If x^2 is divisible by 4, then x is even.
  - Note the "if and only if" structure: all four versions are true. We can, in this uncommon case, abbreviate with "(Let x be an integer.) x is even if and only x^2 is divisible by 4."
- Question π:
  - Original: Hypothesis: If a child is good, then that child will receive lots of presents in late December.
  - Contrapositive: Hypothesis: If a child does not receive lots of presents in late December, then the child is not good.
  - Inverse: Hypothesis: If a child is not good, then that child does not receive lots of presents in late December.
  - Converse: Hypothesis: If a child receives lots of presents in late December, then that child is good.
  - Considering children in Aleppo, Somalia, Uganda, Haiti, and Rwanda should resolve the contrapositive as false. The original is likewise destroyed by those counter-examples. For the inverse and the converse, you might want to consider that Dylann Roof probably received Christmas presents at least once. It appears that all four versions are false.
- Question 4:

- Original: If x is an integer, then $x^3 - x$ is divisible by 3.
- Contrapositive: If $x^3 - x$ is not divisible by 3, then x is not an integer.
- Inverse: If x is not an integer, then $x^3 - x$ is not divisible by 3.
- Converse: If $x^3 - x$ is divisible by 3, then x is an integer.
- Using Sage, one can find out that $f(1.6716998816571609\cdots) = 3$. This was done with the command

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find_root( x^3 - x == 3, -10, 10 )
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  which asks Sage to find a root of the polynomial equation $x^3 - x = 3$, looking in the interval $-10 < x < 10$. Using a hand calculator, you can verify that

$$(1.6716998816571609\cdots)^3 - 1.6716998816571609\cdots \approx 3$$

- After using Sage to find that root, we can see that the converse is false. That root is also a counter-example to the inverse. We will prove the original true as an exercise in this course; Thus the contra-positive is true as well. We return again to the classic structure: the original and the contrapositive are true, while the inverse and the converse are false.

• Question 5:

- Original: If a Stout student's cumulative GPA is below 2.0, then that student is on academic probation.
- Contrapositive: If a Stout student is not on academic probation, then that student's cumulative GPA is 2.0 or higher.
- Inverse: If a Stout student's cumulative GPA is 2.0 or higher, then that student is not on academic probation.
- Converse: If a Stout student is on academic probation, then that student's cumulative GPA is below 2.0.
- While you might not know the academic probation rules at UW Stout, it turns out that the original and contrapositive are true, but the inverse and the converse are false—this is the classic structure. I believe it is the case that if a student has a 2.25 semester GPA, and was on academic probation during the previous semester, then the student will remain on academic probation—even if the cumulative GPA is above 2.0.

• Question 6:

- Original: (Let $f(x) = (x-4)^2 - 1$.) If $x > 5$, then $f(x) > 0$.
- Contrapositive: (Let $f(x) = (x-4)^2 - 1$.) If $f(x) \leq 0$, then $x \leq 5$.
- Inverse: (Let $f(x) = (x-4)^2 - 1$.) If $x \leq 5$, then $f(x) \leq 0$.
- Converse: (Let $f(x) = (x-4)^2 - 1$.) If $f(x) > 0$ then $x > 5$.
- Note the classic structure: the original and the contrapositive are true, while the inverse and the converse are false. To see this, consider $f(-6) = 99$.

• Question 7:

- Original: (Let x and y be integers.) If x is even and y is even, then xy is even.
- Contrapositive: (Let x and y be integers.) If xy is odd, then it is not the case that both x is even and y is even.
- Contrapositive: (Let x and y be integers.) If xy is odd, then either x is odd or y is odd.
- Inverse: (Let x and y be integers.) If it is not the case that both x is even and y is even, then xy is odd.

- Inverse: (Let x and y be integers.) If either x is odd or y is odd, then xy is odd.
- Converse: (Let x and y be integers.) If xy is even, then x is even and y is even.
- Converse: (Let x and y be integers.) If xy is even, then it is not the case that either x is odd or y is odd. (No one would ever speak this way.)
- Note the classic structure: the original and the contrapositive are true, while the inverse and the converse are false. To see this, consider $x = 3$ and $y = 4$.

- Question 8:
  - Original: If $f(x)$ has a local optimum at $x = 5$, then either $f'(5) = 0$ or $f'(5)$ does not exist.
  - Contrapositive: If it is not the case that either $f'(5) = 0$ or $f'(5)$ does not exist, then $f(x)$ does not have a local optimum at $x = 5$.
  - Contrapositive: If $f'(5)$ exists and $f'(5) \neq 0$, then $f(x)$ does not have a local optimum at $x = 5$.
  - Inverse: If $f(x)$ does not have a local optimum at $x = 5$, then it is not the case that either $f'(5) = 0$ or $f'(5)$ does not exist.
  - Inverse: If $f(x)$ does not have a local optimum at $x = 5$, then $f'(5)$ exists and $f'(5) \neq 0$.
  - Converse: If either $f'(5) = 0$ or $f'(5)$ does not exist, then $f(x)$ has a local optimum at $x = 5$.
  - Converse: If it is not the case that both $f'(5)$ exists and $f'(5) \neq 0$, then $f(x)$ has a local optimum at $x = 5$. (This is a plate of verbal spaghetti.)
  - Note the classic structure: the original and the contrapositive are true, while the inverse and the converse are false. To see this, consider $f(x) = (x - 5)^3$. If you graph this, from $3 < x < 7$, then you'll see there is no optimum at $x = 5$. Yet, $f'(x) = 3(x - 5)^2(1)$ thus $f'(5) = 3(0)^2(1) = 0$.

- Question 9:
  - Original: If the cumulative GPA is below 2.0 for the last two semesters, or if the previous semester GPA is below 1.0, then the student gets dismissed.
  - Contrapositive: If the student does not get dismissed, then it is not the case that either the cumulative GPA is below 2.0 for the last two semesters, or the previous semester GPA is below 1.0.
  - Contrapositive: If the student does not get dismissed, then the cumulative GPA is 2.0 or better for the last two semesters, and the previous semester GPA is 1.0 or above.
  - Inverse: If a student has cumulative GPA 2.0 or better for the last two semesters and the previous semester GPA is 1.0 or above, then the student will not get dismissed.
  - Inverse: If it is not the case that a student has either a cumulative GPA below 2.0 for the last two semesters, or the previous semester GPA below 1.0, then the student will not get dismissed.
  - Converse: If a student gets dismissed, then the student either has a cumulative GPA below 2.0 for the last two semesters, or the previous semester GPA was below 1.0.
  - Converse: If a student gets dismissed, then it is not the case that that the student has both a cumulative GPA of 2.0 or better for the last two semesters, and the previous semester GPA above 1.0.
  - Do we now understand why lawyers are so highly paid? By the way, I have no idea for certain if any of this is true or false, but I believe that all four versions are true in the absence of appeals.

- Question 10:
  - Original: If $xy = 0$, then either $x = 0$ or $y = 0$.
  - Contrapositive: If it is not the case that either $x = 0$ or $y = 0$, then $xy \neq 0$.
  - Contrapositive: If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.

- Inverse: If $xy \neq 0$, then $x \neq 0$ and $y \neq 0$.
- Inverse: If $xy \neq 0$, then it is not the case that either $x = 0$ or $y = 0$.
- Converse: If either $x = 0$ or $y = 0$, then $xy = 0$.
- Converse: If its not the case that both $x \neq 0$ and $y \neq 0$, then $xy = 0$.
- Note the "if and only if" structure: all four versions are true. We can, in this uncommon case, abbreviate with "$xy = 0$ if and only either $x = 0$ or $y = 0$."

• Question 11:

- Original: If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$.
- Contrapositive: If $xy < 0$, then it is not the case that both $x \geq 0$ and $y \geq 0$.
- Contrapositive: If $xy < 0$, then either $x < 0$ or $y < 0$.
- Inverse: If either $x < 0$ or $y < 0$, then $xy < 0$.
- Inverse: If it is not the case that both $x \geq 0$ and $y \geq 0$, then $xy < 0$.
- Converse: If $xy \geq 0$, then it is not the case that either $x < 0$ or $y < 0$.
- Converse: If $xy \geq 0$, then $x \geq 0$ and $y \geq 0$.
- Note the classic structure: the original and the contrapositive are true, while the inverse and the converse are false. To see this, consider $x = -2$ and $y = -3$, implying $xy = 6$.

• Question 12:

- Original: (Let $f(x) = (x-4)^2 - 1$.) If $x > 3$ and $x < 5$, then $f(x) < 0$.
- Contrapositive: (Let $f(x) = (x-4)^2 - 1$.) If $f(x) \geq 0$, then it is not the case that both $x > 3$ and $x < 5$.
- Contrapositive: (Let $f(x) = (x-4)^2 - 1$.) If $f(x) \geq 0$, then either $x \leq 3$ or $x \geq 5$.
- Inverse: (Let $f(x) = (x-4)^2 - 1$.) If either $x \leq 3$ or $x \geq 5$, then $f(x) \geq 0$.
- Inverse: (Let $f(x) = (x-4)^2 - 1$.) If it is not the case that both $x > 3$ and $x < 5$, then $f(x) \geq 0$.
- Converse: (Let $f(x) = (x-4)^2 - 1$.) If $f(x) < 0$, then it is not the case that either $x \leq 3$ or $x \geq 5$.
- Converse: (Let $f(x) = (x-4)^2 - 1$.) If $f(x) < 0$, then both $x > 3$ and $x < 5$.
- Note the "if and only if" structure: all four versions are true. We can, in this uncommon case, abbreviate with "(Let $f(x) = (x-4)^2 - 1$.) $f(x) < 0$ if and only if $3 < x < 5$."

• Question 13:

- Original: If x^2 is a negative real number, then x is a purely imaginary number.
- Contrapositive: If x is not a purely imaginary number, then it is not the case that both x^2 is real and x^2 is negative.
- Contrapositive: If x is not a purely imaginary number, then either x^2 is not real or $x^2 \geq 0$.
- Inverse: If either x^2 is not real or $x^2 \geq 0$, then x is not a purely imaginary number.
- Inverse: If it is not the case that both x^2 is real and x^2 is negative, then x is not a purely imaginary number.
- Converse: If x is a purely imaginary number, then x^2 is real and x^2 is negative
- Converse: If x is a purely imaginary number, then it is not the case that either x^2 is not real or $x^2 \geq 0$.
- Note the "if and only if" structure: all four versions are true. We can, in this uncommon case, abbreviate with "x^2 is a negative real number if and only if x is a purely imaginary number."