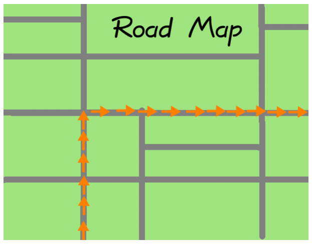
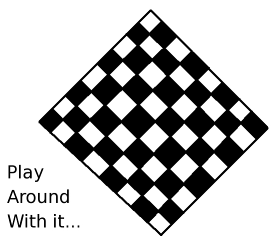
Module 1.1: Introduction to Set Theory



In this chapter we will learn some of the basic operations on sets and what they indicate. Those operations include membership, size, union, intersection, and subset. They are denoted $\in$, #, $\cup$, $\cap$, and $\subseteq$, respectively.

Set theory can be bewildering at first, but if you give it a chance it will eventually become second nature for you.

The first question you might have is "Why do we have set theory?" As you might guess from the fact that set theory appears here in the probability chapter, it is a building block for probability. However, the answer is much deeper. In reality, sets lie at the heart of all branches of mathematics, but that reliance might not yet have been revealed to you. Let us consider something very simple: an ordinary parabola.



1-1-1

What are the solutions to the following degree-two polynomials?

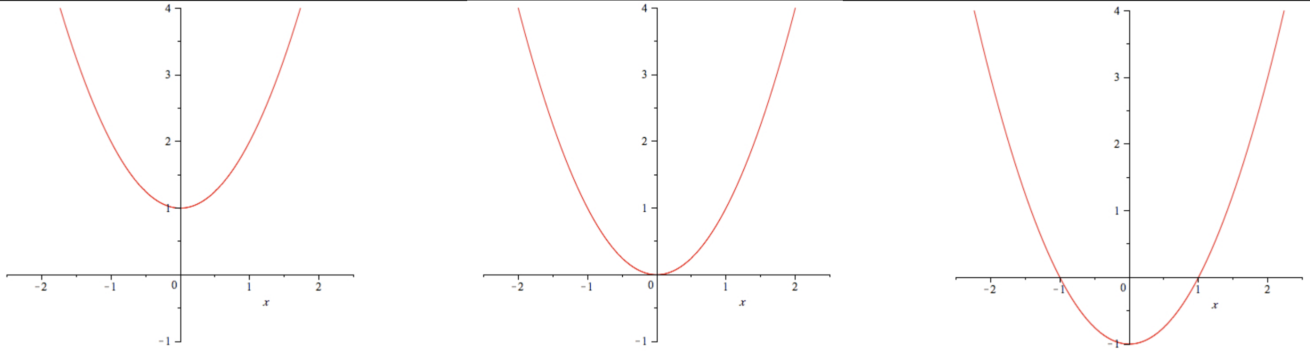
- $x^2 + 1 = 0$ [Answer: No real solutions.]
- $x^2 = 0$ [Answer: The only solution is 0.]
- $x^2 - 1 = 0$ [Answer: The solutions are $+1$ and -1.]

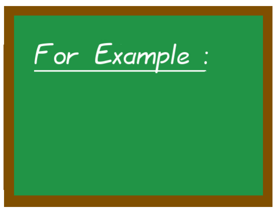
As you can see from the previous box, sometimes the very simple action of solving a quadratic equation has no real answer, one real answer, or two real answers. This can be seen visually from the following three graphs, shown below. The graph on the left has a parabola that never crosses the x-axis; the middle parabola touches the x-axis exactly once; the right parabola crosses the x-axis exactly twice.

For that reason, we expect the polynomial being graphed on the left graph, $x^2 + 1$, to have no roots; similarly, we expect the polynomial being graphed in the middle, x^2, to have one root, and the polynomial on the right $x^2 - 1$, to have two roots.

The question of "What are the roots of this quadratic polynomial?" is fundamentally different than the question of "What is $2 + 3$?" in that the addition of two numbers always produces exactly one answer. However, the question "What are the roots of this polynomial" can produce one answer, several answers, or even zero answers.

This is another reason why we have set theory: to be able to talk about questions that have one answer, several answers, infinitely many answers, or no answers at all.



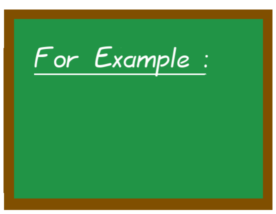


Now let's talk about the set of positive integers less than or equal to five. As a set, we would write them as

$$\{1, 2, 3, 4, 5\}$$

and as you can see, clearly 2 is a member of that set, whereas 6 is not. Thus, we would write $2 \in \{1, 2, 3, 4, 5\}$ and $6 \notin \{1, 2, 3, 4, 5\}$ to express these relationships symbolically. The symbol "$\in$" represents "is a member of," and likewise the symbol $\notin$ represents "is not a member of." This is called *the membership operator*.

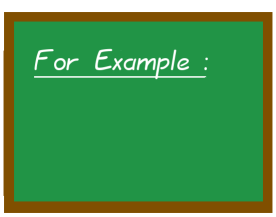
1-1-2



In this notation, the solution sets to the quadratic equations given at the start of this module are $\{\}$, $\{0\}$ and $\{+1, -1\}$. Notice that I could have written the last set as $\{-1, +1\}$. All that matters is that $+1$ is a member of the set, and -1 is a member of that set, and lastly, that nothing else is a member. In fact, order never matters when writing down the members of a set.

It would also be pointless to write down a member of a set more than once. You should never write $\{-1, +1, +1\}$, for example.

1-1-3

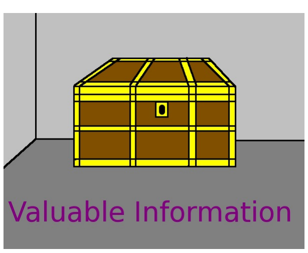


Consider the set of solutions of $x^2 + 1 = 0$. (This is the leftmost graph from the previous page.)

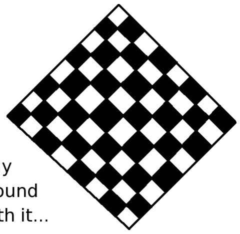
There are no real solutions. Thus, since there are no solutions, we write the set of solutions as $\{\}$.

This is an example of the *empty set*. The circumstance of having no solutions can occur frequently in mathematics, and therefore the empty set occurs more often than one might otherwise assume. The empty set is the set that does not contain any members.

1-1-4



- The symbol "$\in$" is called *the membership operator*, and indicates that some object is in some set. The symbol "$\notin$" indicates that some object is not in some set.
- An *empty set* is a set without any members. Some books use the symbol $\emptyset$ to denote the empty set, but I prefer $\{\}$.



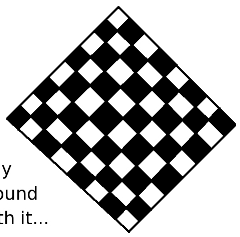
Play
Around
With it...

1-1-5

Some people wish to order pizza:

Patrick likes	Onions, Olives, Green Peppers, Pineapple
Greg likes	Onions, Pepperoni, Mushrooms
Justin likes	Pepperoni, Onions, Olives, Mushrooms
Charles likes	Pepperoni, Onions

- What is the set of people who like Olives? $\{Patrick, Justin\}$.
- What is the set of people who like Pepperoni? $\{Greg, Justin, Charles\}$.
- What is the set of people who like Anchovies? $\{\}$.
- What is the set of people who like Onions? $\{Patrick, Greg, Justin, Charles\}$.



Play
Around
With it...

1-1-6

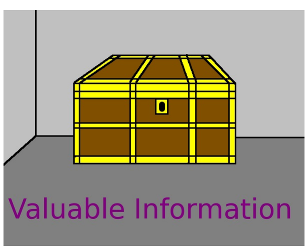
Here's some calisthenics of the mind for you.

- What is the set of US states that do not have the letter "e" in their name?

Hint: (Try this by making a list on a blank piece of paper. It is a good exercise to test your memory. See if you can get all 50 states, then sort them out. It helps to picture the map.)

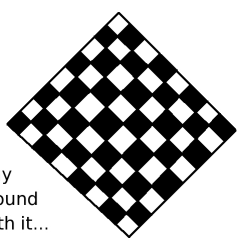
- What is the set of US states that do not have either the letter "e" or the letter "a" in their name?

The answers can be found on Page 53 of this module.



Here are two more vocabulary terms.

- The *roster* of a set is a list of all the elements in a finite set. Obviously this cannot be written for an infinite set. The previous two checkerboard boxes had us writing out the rosters of some sets.
- The *intersection* of two sets $\mathcal{X}$ and $\mathcal{Y}$ is the set containing all the members found in both $\mathcal{X}$ and $\mathcal{Y}$, but nothing more. We write this as $\mathcal{X} \cap \mathcal{Y}$.

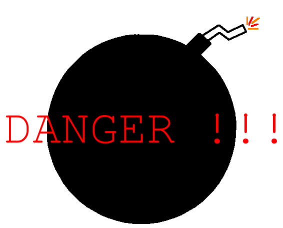


Play
Around
With it...

1-1-7

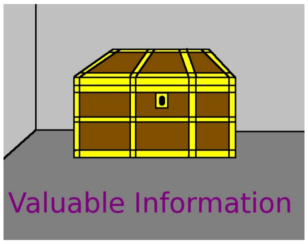
Consider the set $\mathcal{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $\mathcal{E}$ be those numbers in $\mathcal{N}$ that are even. Let $\mathcal{P}$ be those numbers in $\mathcal{N}$ that are prime.

- What is the roster of $\mathcal{P}$? [Answer: $\{2, 3, 5, 7\}$].
- What is the roster of $\mathcal{E}$? [Answer: $\{2, 4, 6, 8, 10\}$].
- What is the roster of $\mathcal{E} \cap \mathcal{P}$? [Answer: $\{2\}$].

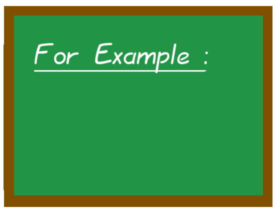


Even though a set may have only one member in it, you are not allowed to drop the braces when expressing the set. In the example above, "$\{2\}$" is the proper way to write the set—you cannot drop the braces and write "2." You must write "$\{2\}$."

The question asked about a set; therefore, the answer must be a set, even if that set contains only one member. Sets with only one member in them are called *singleton* sets.



Very often, it is useful to talk about the size of a set. This will be pivotal when we learn about probability. The size of a set $\mathcal{S}$ is written $\#\mathcal{S}$, and is simply the number of members of the set.



1-1-8

Looking back at the previous checkerboard, we saw

- $\mathcal{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
- $\mathcal{P} = \{2, 3, 5, 7\}$
- $\mathcal{E} = \{2, 4, 6, 8, 10\}$
- $\mathcal{E} \cap \mathcal{P} = \{2\}$

We would write the sizes of those as follows:

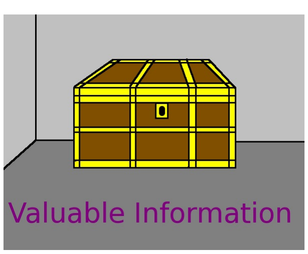
- $\#\mathcal{N} = 10$
- $\#\mathcal{P} = 4$
- $\#\mathcal{E} = 5$
- $\#(\mathcal{E} \cap \mathcal{P}) = 1$
- Just a reminder: since we know $\mathcal{E} \cap \mathcal{P}$ has only one member, we can call it a "singleton" set.

It is also useful to note that the size of the empty set is zero, which would be written $\#\{\} = 0$.

Other textbooks might use different notation for the size of a set.

I have seen $n(\mathcal{A})$, $N(\mathcal{A})$, $|\mathcal{A}|$, card$(\mathcal{A})$, order$(\mathcal{A})$, size$(\mathcal{A})$, as well as my choice for this textbook, $\#\mathcal{A}$.

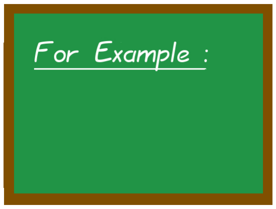
While this is doubtless a source of confusion when moving between textbooks, the good news is that all the other symbols in set theory are universal and in standard usage.



Note, the intersection of two sets is often the empty set. For example, consider the intersection of the set of states which voted for George W. Bush in 2000, and the set of prime numbers. Surely it should be very clear that no item can be members of both sets. When the intersection of two sets is the empty set, we call such sets *disjoint*.

An easier example of disjoint sets is to consider the set of even integers and the set of odd integers. Obviously, there is no integer which is both even and odd simultaneously. Thus, no integer can be found in both sets simultaneously. Therefore, we know the intersection is the empty set, and that the sets are disjoint.

These examples are rather over-simplified, so let us see how these concepts are used in a more practical example.



1-1-9

Consider the stock market data for January 7th, 8th, and 9th of 2002, which can be found in the next box. A trading company has divided the analysis of the 30 Dow Jones Industrial Average companies into an "Industrial & Energy" desk, a "High Technologies" desk, and an "Other Companies" desk. Here, we analyze the "High-Tech" desk.

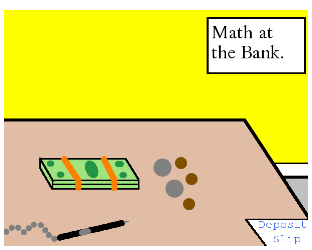
Suppose today is the 9th. We have several basic questions: which stocks closed lower today than they did yesterday? (This means their price dropped between the 8th and the 9th.) Let's call that $\mathcal{B}_1$. Next, which stocks closed lower yesterday than the day before? (That means between the 7th and the 8th.) Let's call that $\mathcal{A}_1$.
As you can see

$$\mathcal{A}_1 = \{T, \quad HPQ, \quad JNJ, \quad PFE, \quad UTX, \quad VZ\}$$

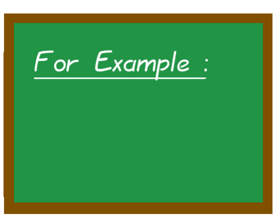
because AT&T, Hewlett Packard, Johnson & Johnson, Pfizer, United Technologies, and Verizon all dropped between the 7th and the 8th. Then, between the 8th and the 9th we have

$$\mathcal{B}_1 = \{T, \quad CSCO, \quad IBM, \quad INTC, \quad JNJ, \quad MSFT, \quad PG, \quad VZ\}$$

because AT&T, Cisco, IBM, Intel, Johnson & Johnson, Microsoft, Proctor & Gamble, and Verizon all were down after the 8th.



Symbol	Company	Day 1 7-Jan-02 Closing	Day 2 8-Jan-02 Closing	Day 3 9-Jan-02 Closing	% change Day 1 → 2	% change Day 2 → 3
T	AT&T	39.91	39.74	38.17	-0.43%	-3.95%
CSCO	Cisco Systems	20.53	20.95	20.85	2.05%	-0.48%
HPQ	Hewlett-Packard	23.02	22.78	23.46	-1.04%	2.99%
IBM	IBM	124.05	124.70	124.49	0.52%	-0.17%
INTC	Intel	35.27	35.58	35.36	0.88%	-0.62%
JNJ	Johnson & Johnson	57.87	57.49	56.91	-0.66%	-1.01%
MRK	Merck	58.10	58.73	58.82	1.08%	0.15%
MSFT	Microsoft	68.56	69.38	68.71	1.20%	-0.97%
PFE	Pfizer	39.70	39.59	40.05	-0.28%	1.16%
PG	Procter & Gamble	77.55	77.76	77.20	0.27%	-0.72%
UTX	United Tech. Corp.	65.85	64.84	65.17	-1.53%	0.51%
VZ	Verizon Comm.	50.30	49.80	48.88	-0.99%	-1.85%

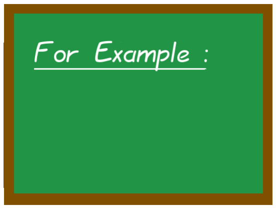


1-1-10

We should be particularly curious about those that are in both $\mathcal{A}_1$ and $\mathcal{B}_1$, as they could be bargains for high-risk investors, or liabilities for low-risk investors. Let's call that $\mathcal{C}_1$.

Now the set $\mathcal{C}_1$ is going to consist only of things that are in both $\mathcal{A}_1$ and $\mathcal{B}_1$, but nothing else. Recall that this is going to be called an intersection. We will write this as $\mathcal{C}_1 = \mathcal{A}_1 \cap \mathcal{B}_1$. Look only at the sets—not at the stock charts—and see that

$$\mathcal{C}_1 = \{T, \quad JNJ, \quad VZ\}$$



1-1-11

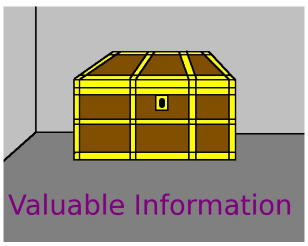
Now, let's further refine this notion. A stock either rises or falls on a given day; it is extremely rare for it to close at the exactly same price as it opened, to the nearest penny. However, some drops are more significant than others. One standard that is actually used in the financial world is to require the stock to drop 1% or more for the drop to be considered important. Let's denote $\mathcal{AA}_1$ as the items in $\mathcal{A}_1$ that dropped by 1% or more, and likewise $\mathcal{BB}_1$ will be defined the same way—those stocks in $\mathcal{B}$ that dropped 1% or more. Therefore, we have

$$\mathcal{AA}_1 = \{HPQ, \quad UTX\}$$

and then also

$$\mathcal{BB}_1 = \{T, \quad JNJ, \quad VZ\}$$

as you can see from the table of prices.



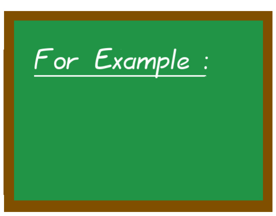
From the above example, we can make the observation that all members of the set $\mathcal{AA}_1$ (stocks that are down 1% or more) are also members of the set $\mathcal{A}_1$ (stocks that are down, in general). From this observation we can say that $\mathcal{AA}_1$ is a *subset* of $\mathcal{A}_1$.

We say that $\mathcal{X}$ is a subset of $\mathcal{Y}$ if and only if every member of $\mathcal{X}$ is a member of $\mathcal{Y}$. We write this as $\mathcal{X} \subseteq \mathcal{Y}$.

Note how $\subseteq$ looks like "C," the first letter of "contains." A good hint is to remember that "subset" means "is contained in."

Also, the $\subseteq$ symbol kind of looks like the $\leq$ symbol. If $\mathcal{X}$ is a subset of $\mathcal{Y}$, then $\mathcal{X}$ cannot have more members than $\mathcal{Y}$. We can even write

$$\text{if } \mathcal{X} \subseteq \mathcal{Y} \text{ then } \#\mathcal{X} \leq \#\mathcal{Y}$$

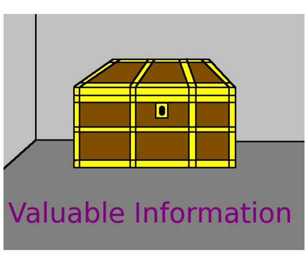


1-1-12

Note that in the above example $\mathcal{CC}_1$ could be defined as those stocks that dropped 1% or more on both days. This would be $\mathcal{AA}_1 \cap \mathcal{BB}_1$, and in this case

$$\mathcal{CC}_1 = \mathcal{AA}_1 \cap \mathcal{BB}_1 = \{HPQ, \quad UTX\} \quad \cap \quad \{T, \quad JNJ, \quad VZ\} = \{\}$$

is the empty set, because $\mathcal{AA}_1$ and $\mathcal{BB}_1$ have nothing in common.

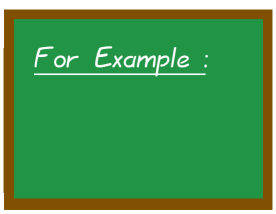


Recall that when two sets have no members in common, they are called *disjoint*. Their intersection will therefore be the empty set.

Verizon just barely missed being in $\mathcal{AA}_1$, having dropped 0.99%, and likewise Microsoft just barely missed being in $\mathcal{BB}_1$, having dropped 0.97%. This can happen. One of the things set theory is good at is helping us talk about problems with multiple solutions, or no solutions. However, one of the flaws of set theory is that there's no notion of "almost." Either an object is a member of a set, or it is not a member of a set—there is no middle ground.

For algebra, to accommodate the notion of "almost," one can extend from the world of equations into the world of inequalities. On the other hand, for set theory, the extension was invented only recently and is called the study of "fuzzy sets." In that subject, one studies sets where membership is not limited to "yes" or "no," but instead, an object can be "somewhat" in a set.

However, in ordinary set theory (which we study in this textbook) "somewhat" is not possible—only "yes" or "no" is possible.



1-1-13

We noted in the previous box that in order for a stock to be down 1% or more, it must first be down. Sure enough,

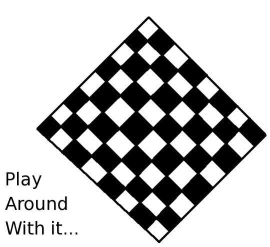
$$\mathcal{AA}_1 = \{HPQ, UTX\} \subseteq \{T, HPQ, JNJ, PFE, UTX, VZ\} = \mathcal{A}_1$$

because each member of $\mathcal{AA}_1$ can be found in $\mathcal{A}_1$.

Likewise,

$$\mathcal{BB}_1 = \{T, JNJ, VZ\} \subseteq \{T, CSCO, IBM, INTC, JNJ, MSFT, PG, VZ\} = \mathcal{B}_1$$

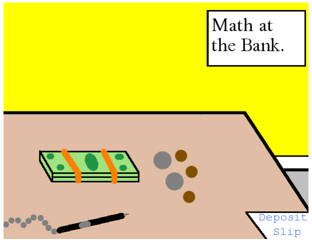
because each member of $\mathcal{BB}_1$ can be found in $\mathcal{B}_1$.



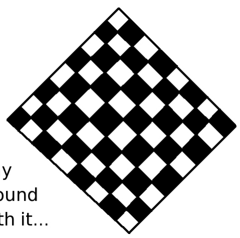
1-1-14

Using the table in the following box, find the following information for the "Industrial & Energy" desk. The answers will be found on Page 53.

- What is the roster of $\mathcal{A}_2$, the set of stocks that fell between day 1 and day 2?
- What is the roster of $\mathcal{AA}_2$, the set of stocks that fell 1% or more between day 1 and day 2?
- What is the roster of $\mathcal{B}_2$, the set of stocks that fell between day 2 and day 3?
- What is the roster of $\mathcal{BB}_2$, the set of stocks that fell 1% or more between day 2 and day 3?
- What is the roster of $\mathcal{C}_2$, the set of stocks that fell on both days?
- What is the roster of $\mathcal{CC}_2$, the set of stocks that fell 1% or more on both days?



Symbol	Company	Day 1 7-Jan-02 Closing	Day 2 8-Jan-02 Closing	Day 3 9-Jan-02 Closing	% change Day 1 $\to$ 2	% change Day 2 $\to$ 3
MMM	3M	115.70	115.05	114.65	-0.56%	-0.35%
AA	Alcoa	38.16	37.34	36.50	-2.15%	-2.25%
BA	Boeing	41.00	40.33	39.90	-1.63%	-1.07%
CAT	Caterpillar	53.15	51.35	51.56	-3.39%	0.41%
CVX	Chevron Corporation	89.72	90.05	89.11	0.37%	-1.04%
DD	DuPont	44.65	44.20	44.10	-1.01%	-0.23%
XOM	ExxonMobil	39.65	39.70	39.24	0.13%	-1.16%
GE	General Electric	39.36	38.95	38.55	-1.04%	-1.03%
DIS	Walt Disney	23.25	22.79	21.80	-1.98%	-4.34%

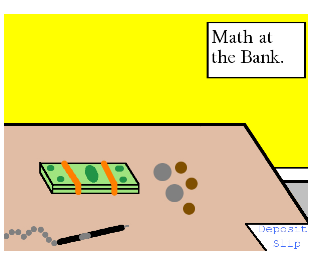


Play
Around
With it...

1-1-15

Using the table in the following box, find the following information for the "Other Companies" desk of the Dow Jones Industrial stocks. The answers will be found on Page 54.

- What is the roster of $\mathcal{A}_3$, the set of stocks that fell between day 1 and day 2?
- What is the roster of $\mathcal{AA}_3$, the set of stocks that fell 1% or more between day 1 and day 2?
- What is the roster of $\mathcal{B}_3$, the set of stocks that fell between day 2 and day 3?
- What is the roster of $\mathcal{BB}_3$, the set of stocks that fell 1% or more between day 2 and day 3?
- What is the roster of $\mathcal{C}_3$, the set of stocks that fell on both days?
- What is the roster of $\mathcal{CC}_3$, the set of stocks that fell 1% or more on both days?



Symbol	Company	Day 1 7-Jan-02 Closing	Day 2 8-Jan-02 Closing	Day 3 9-Jan-02 Closing	% change Day 1 $\to$ 2	% change Day 2 $\to$ 3
AXP	American Express	37.6	37.05	37.15	-1.46%	0.27%
BAC	Bank of America	63.11	62.10	61.95	-1.60%	-0.24%
KO	Coca-Cola	45.22	44.65	44.56	-1.26%	-0.20%
JPM	JPMorgan Chase	38.90	38.60	38.71	-0.77%	0.28%
KFT	Kraft Foods	33.15	33.49	33.60	1.03%	0.33%
MCD	McDonald's	27.20	27.36	26.88	0.59%	-1.75%
HD	The Home Depot	50.59	51.10	50.26	1.01%	-1.64%
TRV	Travelers	42.85	42.42	42.12	-1.00%	-0.71%
WMT	Wal-Mart	57.39	57.84	56.40	0.78%	-2.49%

The concept of "subset" is kind of like "less than or equal to." In fact, the symbol $\subseteq$ is really similar to $\leq$ visually, and there is the relationship with the sizes that we noted on Page 39. However, there is an important difference. For any two numbers x and y, we have either $x \leq y$ or $y \leq x$. Yet, this does not work with sets!

Consider

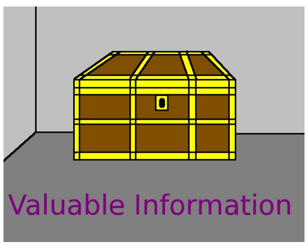
$$\mathcal{A}_3 = \{AXP, BAC, KO, JPM, TRV\} \text{ and } \mathcal{B}_3 = \{BAC, KO, MCD, HD, TRV, WMT\}$$

the stock AXP is in $\mathcal{A}_3$, but not $\mathcal{B}_3$, so there is no way that $\mathcal{A}_3$ could be a subset of $\mathcal{B}_3$. Alternatively, MCD is in $\mathcal{B}_3$, but not $\mathcal{A}_3$, so there is no way that $\mathcal{B}_3$ is a subset of $\mathcal{A}_3$.

This can be abbreviated

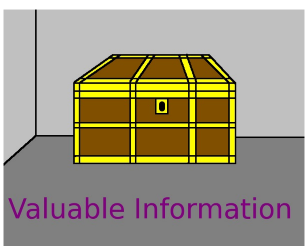
$$\mathcal{A}_3 \not\subseteq \mathcal{B}_3 \text{ and } \mathcal{B}_3 \not\subseteq \mathcal{A}_3$$

where $\not\subseteq$ means "is not contained in" or "is not a subset of."



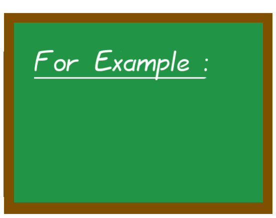
The union of two sets $\mathcal{X}$ and $\mathcal{Y}$ is composed of anything that is part of $\mathcal{X}$ or $\mathcal{Y}$, or both, but nothing else. It is denoted

$$\mathcal{X} \cup \mathcal{Y}$$



When we say an *algebraic expression*, we mean a collection of numbers, variables which eventually represent numbers, and operators including $+, -, \times, \div, \sqrt{\ }, \sqrt[3]{\ }, ^2, ^3, \ldots$ and so on.

Likewise when we say a *set-theoretic expression*, we mean a collection of sets, variables which eventually represent sets, and operators including, $\cup, \cap, \subseteq, \ldots$ and so on. (There's actually one more operator, called "the complement," which we will introduce on Page 63, in the next module.)



\# 1-1-16

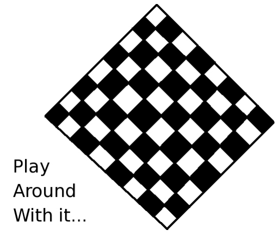
Now we're going to aggregate the reports for the boss. She's interested in stocks that have fallen on both days, thus we will report $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$. However, we have instructions to report any stocks that fall 1% or more on either day. So we would also want to report $\mathcal{AA}_1$, $\mathcal{AA}_2$, and $\mathcal{AA}_3$, as well as $\mathcal{BB}_1$, $\mathcal{BB}_2$, and $\mathcal{BB}_3$. What we're going to use is the union operation. The set-theory equation

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$$

means that we take anything listed in either $\mathcal{C}_1$, $\mathcal{C}_2$, or $\mathcal{C}_3$. In that case,

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 = \{T, JNJ, VZ, MMM, AA, BA, DD, GE, DIS, BAC, KO, TRV\}$$

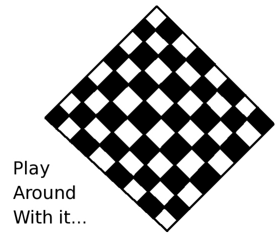
Now you shall find $\mathcal{AA}$ and $\mathcal{BB}$ yourself, in the next box.



1-1-17

Now we're going to compute the union of two more sets. The answers can be found on Page 54.

- What is the roster of $\mathcal{AA} = \mathcal{AA}_1 \cup \mathcal{AA}_2 \cup \mathcal{AA}_3$?
- What is the roster of $\mathcal{BB} = \mathcal{BB}_1 \cup \mathcal{BB}_2 \cup \mathcal{BB}_3$?



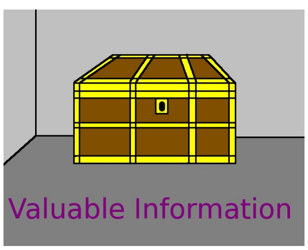
1-1-18

Suppose the decision is to sell anything that is down 1% or more on either day, or something that was down both days.

- What set-theory equation would describe that? [Answer: $\mathcal{C} \cup \mathcal{AA} \cup \mathcal{BB}$.]
- What is the roster of that set?

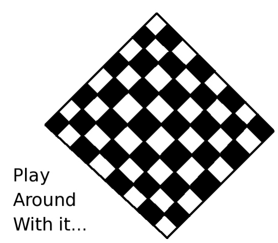
$$[\text{Answer: } \{AA, AXP, BA, BAC, CAT, CVX, DD, GE, DIS, HD, HPQ, JNJ, \\ KO, MCD, MMM, T, TRV, UTX, XOM, VZ, WMT\}.]$$

It is not necessary to alphabetize the members of a set, but be sure not to list any twice. Sometimes the easiest way to avoid leaving a member out, or duplicating a member, is to alphabetize them.



The meaning of "=" for sets is very simple: two sets are equal if and only if they have precisely the same members.

That's another reason that order never matters in sets. If two sets have the same membership, then they are equal regardless of the ordering. Contrasingly, if you find a member in one set, which is not found in the other set, then you have to say that the sets are not equal, or use the $\neq$ symbol.



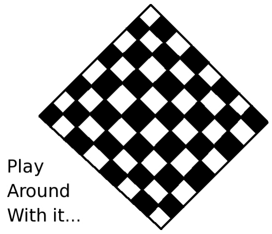
1-1-19

Given the following sets,

$$\mathcal{A} = \{-1, 1, 3, 5\} \qquad \mathcal{B} = \{-1, 0, 1, 2, 3, 4, 5\} \qquad \mathcal{C} = \{1, -1, 5, 3\}$$

decide which questions below are true or false, but do not peak at the answer until you are finished with the problem! (The answers can be found on Page 54.)

1. $A = B$
2. $B \neq C$
3. $A = C$
4. $A \subseteq B$
5. $B \subseteq C$
6. $C \not\subseteq B$
7. $C \subseteq A$
8. $B \subseteq A$

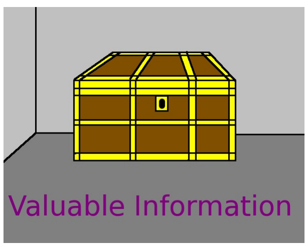


Give an example of a subset of $\mathcal{B}$ that is not a subset of A, knowing that $\mathcal{A} = \{13, 17, 19, 23\}$ and $\mathcal{B} = \{13, 17, 11\}$.

[Answer: Examples include $\{11, 17\}$; also $\{11, 13\}$; as well as $\mathcal{B}$ itself.]

Note that the inclusion of 11 in your answer is necessary, as this is the only member of $\mathcal{B}$ that is not a member of $\mathcal{A}$.

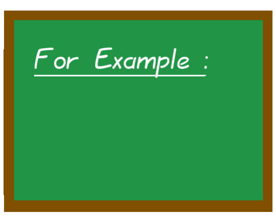
1-1-20



The symbol $\cap$ for intersection, and $\cup$ for union are confusing for some students. An old trick is to notice that $\cup$ looks like a "u" and there's no "u" in "intersection," while there is a "u" in "union." I have another favorite memory hook:

Because the union is composed of things *either* in $\mathcal{A}$ or $\mathcal{B}$ and the intersection is composed of things that are *both* in $\mathcal{A}$ and $\mathcal{B}$, then it is clear that the union is bigger. By that we mean that the intersection is never larger than the union. In fact, the intersection is almost always smaller than the union (sometimes much smaller).

Now the other memory hook is to imagine a beer mug. If I hold it like $\cap$, then all the beer will spill on to the floor, and very little beer will remain in the mug—just a few drops adhering to the inside walls of the mug; but if I hold it like $\cup$, then I can put a lot of beer into the mug. Thus $\cup$ represents the larger set, that is to say, the union. Then $\cap$ represents the smaller set, that is to say, the intersection.



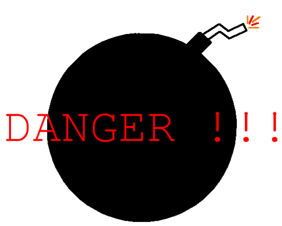
Suppose one has $\mathcal{A} = \{1, 4, 9\}$ and also $\mathcal{B} = \{1, 3, 5, 7, 9\}$. What is $\mathcal{A} \cap \mathcal{B}$? What is $\mathcal{A} \cup \mathcal{B}$?

The intersection (that's $\mathcal{A} \cap \mathcal{B}$) is the set of members found in both $\mathcal{A}$ and $\mathcal{B}$. So then that's clearly $\{1, 9\}$.

Next, the union (that's $\mathcal{A} \cup \mathcal{B}$) is anything you can find in either set. Thus we have

$$\mathcal{A} \cup \mathcal{B} = \{1, 3, 4, 5, 7, 9\}$$

1-1-21



Notice, we do not write

$$WRONG \rightarrow \qquad \mathcal{A} \cup \mathcal{B} = \{1, 1, 3, 4, 5, 7, 9, 9\} \qquad \leftarrow WRONG$$

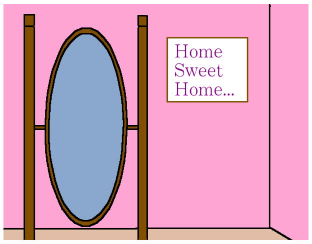
even though 1 and 9 are found both in $\mathcal{A}$ and $\mathcal{B}$. We only write 1 and 9 once each. This is not just a mathematical convention, as it turns out.

The reason for this rule is that if we write it correctly, as

$$\mathcal{A} \cup \mathcal{B} = \{1, 3, 4, 5, 7, 9\}$$

then we can see at a glance that there are six integers in $\mathcal{A} \cup \mathcal{B}$. If we wrote it with duplication, we might imagine that there are eight integers in $\mathcal{A} \cup \mathcal{B}$, which is wrong.

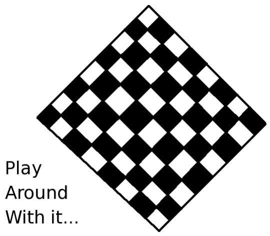
After all, it is clear that the question "How many integers are in either $\mathcal{A}$, $\mathcal{B}$, or both?" has as its answer "six." Therefore, it is wrong to write the set with the 1 and 9 repeated, because there should be six things between the { and the }.



A Pause for Reflection...
As you do the next two checkerboards, think about under what conditions, for a set $\mathcal{A}$ and a set $\mathcal{B}$, can the intersection equal the union? More precisely, when can

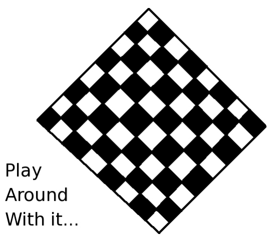
$$\mathcal{A} \cup \mathcal{B} = \mathcal{A} \cap \mathcal{B}$$

We will answer this question immediately after the next two checkerboards.



1-1-22

Suppose $\mathcal{X} = \{1, 10, 100, 1000\}$ and $\mathcal{Y} = \{1, 10, 100\}$. What is $\mathcal{X} \cap \mathcal{Y}$? How about $\mathcal{X} \cup \mathcal{Y}$?
The answers can be found on Page 54.



1-1-23

Consider $\mathcal{A} = \{1, 2, 3, 4\}$ and $\mathcal{B} = \{4, 3, 2, 1\}$.

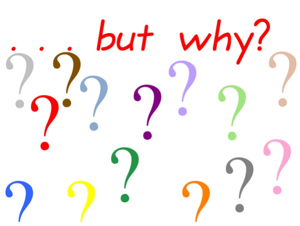
- What is $\mathcal{A} \cup \mathcal{B}$?
- What is $\mathcal{A} \cap \mathcal{B}$?

Likewise, the answers can be found on Page 54.

While the previous box might seem trivial, it goes to show that

$$\text{if } \mathcal{A} = \mathcal{B} \text{ then } (\mathcal{A} \cup \mathcal{B}) = (\mathcal{A} \cap \mathcal{B})$$

What is far less obvious, is that this is the only way that the union can equal the intersection. Put another way, if the two original sets are different, then for sure, the union and the intersection are different.
We will prove this in the next box.



Our goal is to prove that if two sets are not equal, then their union cannot equal their intersection. In other words,

$$\text{if } \mathcal{A} \neq \mathcal{B} \text{ then } (\mathcal{A} \cup \mathcal{B}) \neq (\mathcal{A} \cap \mathcal{B})$$

To see that, think about what it means for $\mathcal{A} \neq \mathcal{B}$. That means either there is something in $\mathcal{A}$ that is not in $\mathcal{B}$, or alternatively, there is something in $\mathcal{B}$ not in $\mathcal{A}$. Call that thing x. Surely,

$$x \in (\mathcal{A} \cup \mathcal{B})$$

and even more obviously,

$$x \notin (\mathcal{A} \cap \mathcal{B})$$

Yet now that we've identified a member in $\mathcal{A} \cap \mathcal{B}$ that is not in $\mathcal{A} \cup \mathcal{B}$ we've shown that the union and intersection are definitely not equal sets.

Perhaps when doing a study abroad in the United Kingdom, you meet a group of 10 students. Their basic data can be found below. We're going to analyze some relationships among them through set theory, in the course of a few checkerboard boxes.

Name	Major	Origin	Name	Major	Origin
Alistair	Music	Canada	Geoffrey	Philosophy	England
Benjamin	Economics	England	Hugh	Biology	Wales
Chauncey	Philosophy	Wales	Randall	Economics & Psychology	Scotland
Dominic	Economics	Wales	Percy	Philosophy	England
Edward	Philosophy & Economics	England	Sebastian	Economics	England

The questions about these students follow in the next few boxes.

Play Around With it...

1-1-24

Referring to the data in the previous box, let $\mathcal{E}$ represent those students studying economics; let $\mathcal{P}$ represents those students studying philosophy. Let $\mathcal{W}$ represent those students from Wales, and let $\mathcal{A}$ represent those students from England. (Note that $\mathcal{E}$ is already taken, so $\mathcal{A}$ for Albion makes sense as a second choice.)

You can abbreviate students' names by the first letter of their name. Also, to be super clear, if someone has a double major then they fully count as a member of both majors.

1. List the roster of $\mathcal{A} \cap \mathcal{E}$.
2. List the roster of $\mathcal{P} \cup \mathcal{E}$.
3. List the roster of $\mathcal{W} \cap \mathcal{P}$.
4. List the roster of $\mathcal{A} \cup \mathcal{P}$.

The answers are found on Page 55.

Play Around With it...

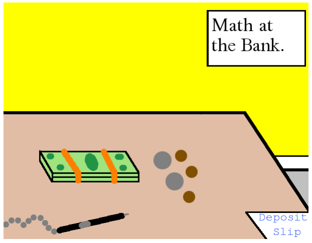
1-1-25

Continuing with the data from the previous box, in a few words, describe the idea of the following sets. For example, $\mathcal{A} \cap \mathcal{E}$ can be described as "the set of English Economists," or $\mathcal{P} \cup \mathcal{W}$ can be described as "students who are either Welsh or studying Philosophy."

5. Describe $\mathcal{P} \cap \mathcal{A}$
6. Describe $\mathcal{W} \cup \mathcal{E}$
7. Describe $\mathcal{W} \cap \mathcal{E}$
8. Describe $\mathcal{A} \cap \mathcal{P}$

The answers are found on Page 55.

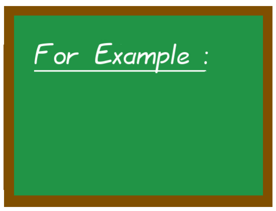
As the previous problem shows, set theory is like a language. It is good to practice translating to and from a new language when you are first learning it.



Suppose a large company needs to hire twenty interns, and they get a thousand applications. There has to be a way to sort through all of those without too much human intervention. The solution would be a database. Nearly all databases in use today accept queries written in `SQL` ("structured query language") or one of its variants, such as `mySQL`.

In many ways, `SQL` is just a slightly more formal and wordy version of set theory, using computer commands instead of set theory symbols. Moreover, it is openly inspired by and built upon set theory.

That means it is useful for anyone, in this era of "Big Data," to learn set theory.



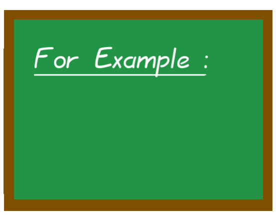
1-1-26

Imagine a company that wishes to hire interns. Define the following sets

- $\mathcal{E}$ = Applicants who know MS-Excel.
- $\mathcal{W}$ = Applicants who know MS-Word.
- $\mathcal{A}$ = Applicants who know MS-Access.
- $\mathcal{P}$ = Applicants who know MS-Powerpoint.
- $\mathcal{G}$ = Applicants who have a GPA above 3.0.

We can note the following translations:

- Applicants who know MS-Word and MS-Powerpoint: $\mathcal{W} \cap \mathcal{P}$.
- Applicants who know MS-Powerpoint or MS-Excel: $\mathcal{P} \cup \mathcal{E}$.



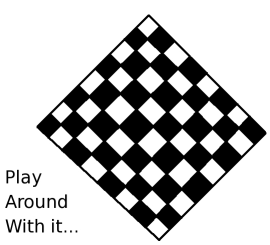
1-1-27

By the way, we're allowed to use parentheses in set theory. Continuing with the previous example, if a manager wants applicants who know MS-Excel and MS-Word, as well as at least one of the two remaining Microsoft products, then we can write

$$(\mathcal{E} \cap \mathcal{W} \cap \mathcal{A}) \cup (\mathcal{E} \cap \mathcal{W} \cap \mathcal{P})$$

or equivalently

$$\mathcal{E} \cap \mathcal{W} \cap (\mathcal{A} \cup \mathcal{P})$$

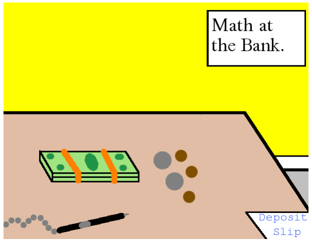


1-1-28

Continuing with the previous two boxes, how would you write

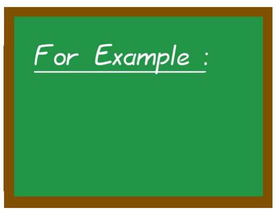
- Applicants who know any one of the four products.
- Applicants who know all four products.
- Applicants who know MS-Powerpoint, and either MS-Word or MS-Excel?

The answer is given on Page 55.



You can also use the operations of set theory when doing internet searches on Google or other search engines, if you know the proper and specific commands. Regrettably, these commands change from time to time and are therefore not suitable for inclusion in textbook. The examples would become obsolete too quickly.

Both for learning to use tools like `SQL` and for performing better Google searches, it is useful to understand set theory.



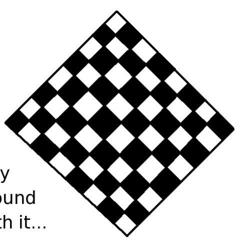
1-1-29

Suppose you are helping a friend with his homework in a different course, and he has the following problem. Let S_6 be the set of real numbers x such that $f(x) = 6$, where $f(x) = 2 + x^2$. What is S_6?

Well, you can probably see that we're being asked to solve

$$\begin{aligned} 6 &= f(x) \\ 6 &= 2 + x^2 \\ 4 &= x^2 \\ \pm 2 &= x \end{aligned}$$

Therefore, you should write $S_6 = \{-2, 2\}$ as your final answer.

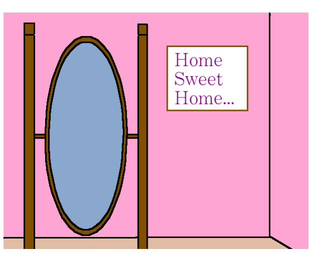


Play
Around
With it...

1-1-30

Continuing with the problem of the previous box,

- If S_3 is the set of real numbers x such that $f(x) = 3$, then what is S_3?
- If S_{11} is the set of real numbers x such that $f(x) = 11$, then what is S_{11}?
- If S_2 is the set of real numbers x such that $f(x) = 2$, then what is S_2?
- If S_1 is the set of real numbers x such that $f(x) = 1$, then what is S_1?
- The answers can be found on Page 55.



A Pause for Reflection...

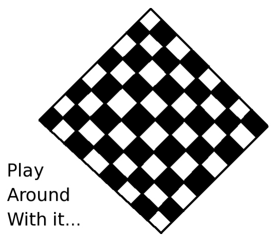
Now, I want you to first check your answers to the previous checkerboard. It is very important that you do that before reading this box...

...Have you checked your answer? I wasn't joking; it is really important!

Now that you have checked your answer, suppose someone asks you the following: "What is the difference between {} and {0}?" Do those two collections of mathematical symbols mean the same thing? If they mean different things, then what is the distinction?

The way that I would answer the previous box is to note some traits:

- The set $\{0\}$ has 1 member.
- The set $\{\}$ has 0 members.
- The set $\{0\}$ is the set of solutions to $2 = 2 + x^2$.
- The set $\{\}$ is the set of (real number) solutions to $1 = 2 + x^2$, or more plainly, $-1 = x^2$. Since no real number can satisfy that equation, the solution set is the empty set.
- In conclusion, $\{0\}$ and $\{\}$ are two very distinct things.



1-1-31

This might be a bit hard, but can you imagine under exactly what circumstances $\mathcal{S} \cup \mathcal{T} = T$? How about when $\mathcal{S} \cap \mathcal{T} = S$? The answer will be given on Page 56. Don't look there, but give yourself a few minutes to think about this.

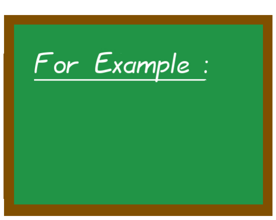
If you can think of an answer to either or to both, then that's great. If not, don't sweat it—this is an abstract notion.

As we said earlier, if $\mathcal{A}$ is a subset of $\mathcal{B}$ then everything that is in $\mathcal{A}$ must also be found in $\mathcal{B}$. Therefore, if you ever need to prove that some set $\mathcal{N}$ is not a subset of $\mathcal{B}$, then all you need do is produce one member of $\mathcal{N}$ that is not in $\mathcal{B}$.

Now I'm going to claim $\{\} \subseteq \mathcal{S}$ for any set $\mathcal{S}$. Do you believe me? If so, then good. However, if you disagree, then you would be compelled to produce something inside of $\{\}$ that is not in $\mathcal{S}$.

Yet, you cannot do that. Why can you not do that? Because $\{\}$ does not have any members of any kind. In conclusion, you must surely agree that $\{\} \subseteq \mathcal{S}$ for every set $\mathcal{S}$.

We have just shown that the empty set is a subset of each and every set in all of set theory.



1-1-32

Consider the set $\{1, 0, -1\}$. What subsets of it exist?

We just learned that $\{\}$ is a subset of every set, so we'll start with that one. How about the singleton sets $\{1\}$, $\{0\}$, and $\{-1\}$? Surely they are subsets!

How about the subsets with two members? If a subset has two members, since there are three to start with, we'd be leaving one member out. Leaving out 1 gives $\{0, -1\}$; leaving out 0 gives $\{1, -1\}$; and finally leaving out -1 gives $\{0, 1\}$.

The last one I'm going to tell you about will probably bother you a bit. In any case, it bothers me. A mathematician (including me) must say that

$$\{1, 0, -1\} \subseteq \{1, 0, -1\}$$

which seems really odd. We must accept this strange statement because everything on the left can be found on the right. That was the definition of subset (consult Page 39 if you like), so we have to go with it.

That makes 8 subsets! Listing them all together, we have

$$\{\}\,;\{1\}\,;\{0\}\,;\{-1\}\,;\{-1,0\}\,;\{0,1\}\,;\{-1,1\}\,;\{-1,0,1\}$$

Apparently I'm not the only one annoyed with the fact that

$$\{-1,0,1\} \subseteq \{-1,0,1\}$$

because mathematicians have a term called *a proper subset*. A proper subset of $\mathcal{S}$ is a set that is firstly, a subset of $\mathcal{S}$, and secondly, is not equal to $\mathcal{S}$. This is not a very important notion for us.

You can abbreviate "$\mathcal{A}$ is a proper subset of $\mathcal{B}$" with $\mathcal{A} \subsetneq \mathcal{B}$.

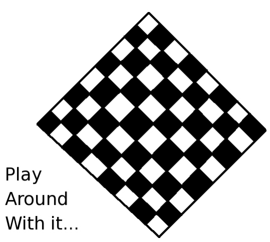
Similarly, some people are bothered by the fact

$$\{\} \subseteq \{-1,0,1\}$$

and mathematicians have a vocabulary term to cover this too. A *non-trivial subset* of $\mathcal{S}$ is a set that is firstly, a subset of $\mathcal{S}$, and secondly, is not empty. Again, this is not very important.

Why, then, am I bothering you with the notions of a *non-trivial subset* and a *proper subset*?

The reason is that when you are asked to list (or count) the subsets of some set, you must always remember that the set itself, as well as the empty set, are in that list.

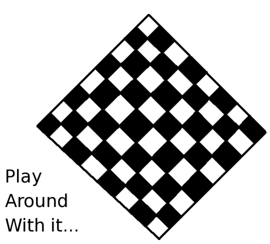


\# 1-1-33

What are all the subsets of

- $\{A,B,C\}$? [Answer: $\{\}$; $\{A\}$; $\{B\}$; $\{C\}$; $\{A,B\}$; $\{B,C\}$; $\{A,C\}$; $\{A,B,C\}$.]
- $\{X,Y,Z\}$? [Answer: $\{\}$; $\{X\}$; $\{Y\}$; $\{Z\}$; $\{X,Y\}$; $\{Y,Z\}$; $\{X,Z\}$; $\{X,Y,Z\}$.]

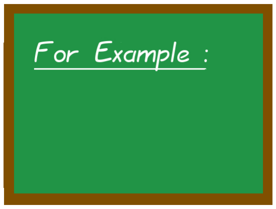
At this point, you have probably figured out that every 3-member set has 8 possible subsets.



\# 1-1-34

What are ...

- ... all the proper subsets of $\{X,Y,Z\}$?
  [Answer: $\{\}$; $\{X\}$; $\{Y\}$; $\{Z\}$; $\{X,Y\}$; $\{Y,Z\}$; $\{X,Z\}$.]
- ... all the non-trivial subsets of $\{X,Y,Z\}$?
  [Answer: $\{X\}$; $\{Y\}$; $\{Z\}$; $\{X,Y\}$; $\{Y,Z\}$; $\{X,Z\}$; $\{X,Y,Z\}$.]
- ... all the proper non-trivial subsets of $\{X,Y,Z\}$?
  [Answer: $\{X\}$; $\{Y\}$; $\{Z\}$; $\{X,Y\}$; $\{Y,Z\}$; $\{X,Z\}$.]



1-1-35

Now let's consider all possible subsets of $\{-1, 0, 1, 2\}$.

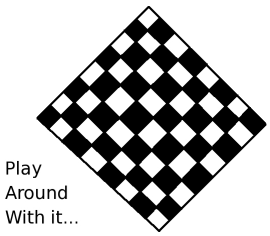
To tackle this problem, all we need do is realize that either 2 is inside any particular subset, or it isn't. If it isn't, then it is a subset of $\{-1, 0, 1\}$ and we worked that out in the previous example box! So we recopy

$$\{\}\,;\{1\}\,;\{0\}\,;\{-1\}\,;\{-1,0\}\,;\{0,1\}\,;\{-1,1\}\,;\{-1,0,1\}$$

and then to take care of those that include 2, we can just attach 2 to each of the subsets we just wrote down. Then, we'd get

$$\{2\}\,;\{1,2\}\,;\{0,2\}\,;\{-1,2\}\,;\{-1,0,2\}\,;\{0,1,2\}\,;\{-1,1,2\}\,;\{-1,0,1,2\}$$

and that's all that can exist. I count 16 possible subsets.

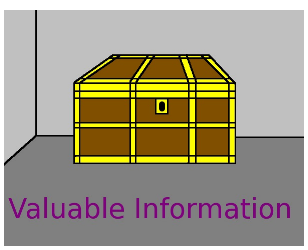


1-1-36

What are all the possible subsets of

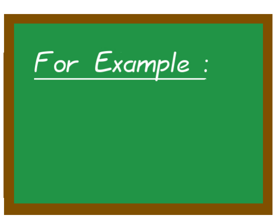
- $\{R, S\}$? [Answer: $\{\}$; $\{R\}$; $\{S\}$; $\{R, S\}$.]
- $\{R, S, T\}$? [Answer: $\{\}$; $\{R\}$; $\{S\}$; $\{T\}$; $\{R, S\}$; $\{S, T\}$; $\{R, T\}$; $\{R, S, T\}$.]
- $\{R, S, T, U\}$?

$$\left[Answer: \begin{array}{c} \{\}\,;\{R\}\,;\{S\}\,;\{T\}\,;\{R,S\}\,;\{S,T\}\,;\{R,T\}\,;\{R,S,T\}\,;\{U\}\,; \\ \{R,U\}\,;\{S,U\}\,;\{T,U\}\,;\{R,S,U\}\,;\{S,T,U\}\,;\{R,T,U\}\,;\{R,S,T,U\} \end{array} . \right]$$



If a set $\mathcal{S}$ has n things in it, then it has 2^n possible subsets.

That collection of 2^n possible subsets of $\mathcal{S}$ includes the empty set $\{\}$, as well as $\mathcal{S}$ itself.



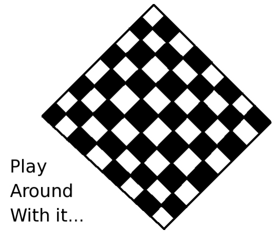
1-1-37

Suppose a major study is going to be performed to determine which patients are likely to survive lymphoma (a type of cancer) and which are likely to not survive. (With enough people in the study, you can even find the probabilities of survival, but we're not ready for that just yet.)

Two medical doctors are consulted, and one makes a list of 6 major factors: gender, obese/non-obese, smoking/non-smoking, drinking/non-drinking, diabetic/non-diabetic, and (lastly) vegetarian/non-vegetarian. The other doctor thinks that smoking and vegetarian-status are irrelevant but recommends the other 4 factors. If we write gender as male/female, then each patient has (as his or her description) a subset of the following factors:

$$\{\text{male, obese, smoker, drinker, diabetic, vegetarian}\}$$

For example, if Patient #1 is female, non-obese, a smoker, a non-drinker, non-diabetic, and a vegetarian, then she would be represented by {smoker, vegetarian}. If Patient #2 is male, obese, a non-smoker, a non-drinker, and diabetic, but not a vegetarian, then he would be represented by {male, obese, diabetic}. If Patient #3 is female, non-obese, a non-smoker, a non-drinker, not diabetic, and not a vegetarian, then she is represented by {}.

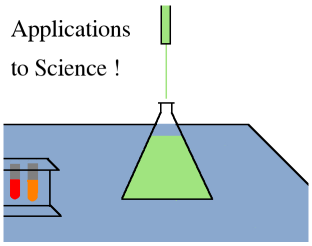


1-1-38

Working from the previous box...

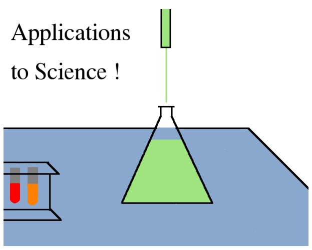
- How many subsets are possible, if using all 6 factors? [Answer: $2^6 = 64$.]
- How about if we drop to 4 factors, by excluding smoking and vegetarian-status? [Answer: $2^4 = 16$.]

Note, this is a serious matter. If we require 100 patients per subset to ensure statistical validity, then we may have a problem, in that 1600 patients might be available, but 6400 patients might not be. Perhaps only 400 are available.



As you surely noticed in the previous box, we neglected to include the transgender community. That can be an issue for the design of a medical study.

For example, consider the possibility of using the smaller-sized experiment, starting with 1600 patients, 800 male and 800 female. Is it feasible to locate 800 transgender persons, and enroll them all in the same medical study?



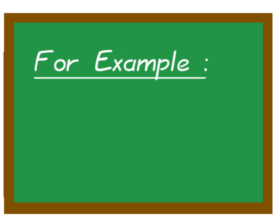
Back in October of 2009, I taught a course *Experimental Design* at the International University of Monaco, as part of their doctoral program. The sorts of factors we analyzed in the previous box are called *boolean factors*. Each patient either has the factor, or does not have the factor.

An experiment working off of this model is called "a 2^N experimental design." When an experiment has a large number of boolean factors (e.g. 12 or 25), then there may be too many possible combinations. In these cases

$$2^{12} = 4096 \text{ and } 2^{25} = 33,554,432$$

and normally you want around 10-1000 people for each combination. On the other hand, studies with 40,000 or more subjects are phenomenally expensive and very rare.

Therefore, any given experiment cannot, if accuracy is desired, analyze more than a relatively limited number of factors. An excellent book on this subject (with an outstandingly humorous opening chapter) is *Design of Comparative Experiments*, by R. A. Bailey, published by the Cambridge University Press.



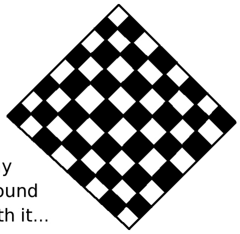
1-1-39

Talking about cancer is depressing, so I'd like to talk about pizza now. Let's consider the possibility of selling pizza with any combination of the following toppings:

$$\{\text{extra cheese, pepperoni, meatballs, green peppers, onions, mushrooms, anchovies}\}$$

How many possible pizzas can be made under this system? Any given pizza represents a subset of the set of toppings. For example, a plain pizza is the empty set. While it would most certainly be a grotesque pizza, "everything on it" would be the improper subset. Thus, every possible pizza produces a subset, and every possible subset produces a pizza.

How many different ways are there to do this? One for every subset possible! How many subsets are there? There are $2^7 = 128$.



Play Around With it...

1-1-40

What if there are

- 14 possible toppings? [Answer: $2^{14} = 16,384$.]
- 8 possible toppings? [Answer: $2^8 = 256$.]

You have now completed this module. All that remains is a listing of the answers to a few checkerboards from earlier in the module.

On Page 36, I asked you some questions about US States.

- What is the set of US states that do not have the letter "e" in their name? [Answer: { Alabama, Alaska, Arizona, Arkansas, California, Colorado, Florida, Hawai'i, Idaho, Indiana, Illinois, Iowa, Kansas, Louisiana, Maryland, Michigan, Mississippi, Missouri, Montana, North Carolina, North Dakota, Ohio, Oklahoma, South Carolina, South Dakota, Utah, Virginia, Washington, Wisconsin, Wyoming }.]
- What is the set of US states that do not have either the letter "e" or the letter "a" in their name? [Answer: {Illinois, Mississippi, Missouri, Ohio, Wisconsin, Wyoming}.]

Here are the answers to the questions from the checkerboard box on Page 40.

- The set of stocks that fell between day 1 and day 2?
  [Answer: $\mathcal{A}_2 = \{MMM, AA, BA, CAT, DD, GE, DIS\}$.]
- The set of stocks that fell 1% or more between day 1 and day 2?
  [Answer: $\mathcal{AA}_2 = \{AA, BA, CAT, DD, GE, DIS\}$.]
- The set of stocks that fell between day 2 and day 3?
  [Answer: $\mathcal{B}_2 = \{MMM, AA, BA, CVX, DD, XOM, GE, DIS\}$.]
- The set of stocks that fell 1% or more between day 2 and day 3?
  [Answer: $\mathcal{BB}_2 = \{AA, BA, CVX, XOM, GE, DIS\}$.]
- The set of stocks that fell both days?
  [Answer: $\mathcal{C}_2 = \{MMM, AA, BA, DD, GE, DIS\}$.]
- The set of stocks that fell 1% or more on both days?
  [Answer: $\mathcal{CC}_2 = \{AA, BA, GE, DIS\}$.]

Here are the answers to the questions from the checkerboard box on Page 41.

- The set of stocks that fell between day 1 and day 2:
  [Answer: $\mathcal{A}_3 = \{AXP, BAC, KO, JPM, TRV\}$.]
- The set of stocks that fell 1% or more between day 1 and day 2:
  [Answer: $\mathcal{AA}_3 = \{AXP, BAC, KO, TRV\}$.]
- The set of stocks that fell between day 2 and day 3:
  [Answer: $\mathcal{B}_3 = \{BAC, KO, MCD, HD, TRV, WMT\}$.]
- The set of stocks that fell 1% or more between day 2 and day 3:
  [Answer: $\mathcal{BB}_3 = \{MCD, HD, WMT\}$.]
- The set of stocks that fell both days: [Answer: $\mathcal{C}_3 = \{BAC, KO, TRV\}$.]
- The set of stocks that fell 1% or more on both days: [Answer: $\mathcal{CC}_3 = \{\}$.]

Here are the answers to the checkerboard box on Page 43.

$$\mathcal{AA} = \mathcal{AA}_1 \cup \mathcal{AA}_2 \cup \mathcal{AA}_3 = \{AA, BA, CAT, DD, GE, DIS, HPQ, UTX, AXP, BAC, KO, TRV\}$$

and also

$$\mathcal{BB} = \mathcal{BB}_1 \cup \mathcal{BB}_2 \cup \mathcal{BB}_3 = \{AA, BA, CVX, XOM, GE, DIS, T, JNJ, VZ, MCD, HD, WMT\}$$

Here are the anewers to the questions about $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, found on Page 43.
[Answer: #1=False; #2=True; #3=True; #4=True;
#5=False; #6=False; #7=True; #8=False.]

These are the answers to the two checkerboard boxes beginning on Page 45.

- $\mathcal{X} \cap \mathcal{Y} = \{1, 10, 100\}$
- $\mathcal{X} \cup \mathcal{Y} = \{1, 10, 100, 1000\}$
- $\mathcal{A} \cap \mathcal{B} = \{1, 2, 3, 4\}$
- $\mathcal{A} \cup \mathcal{B} = \{1, 2, 3, 4\}$

Here are the solutions to the questions about the 10 students from the United Kingdom, which started on Page 46.

- #1 is {B, E, S}.
- #2 is {B, C, D, E, G, R, P, S}.
- #3 is {C}.
- #4 is {B, C, E, G, P, S}.

Continuing with the questions about the 10 students from the United Kingdom, which started on Page 46, the answers for the next four are going to vary from student to student. Different people will have different ways of phrasing the answers.

Do not focus on getting your wording to be the same as my wording. What is important is the idea that is expressed for each answer.

- #5 is "English Philosophers."
- #6 is "students either from Wales or studying Economics."
- #7 is "Welsh Economists."
- #8 would be the same as # 1 because all I did was interchange $\mathcal{P}$ and $\mathcal{A}$.

Here are the answers to the question about processing an excessive number of internship applicants, using set theory notation, from Page 47.

- $\mathcal{A} \cup \mathcal{P} \cup \mathcal{W} \cup \mathcal{E}$
- $\mathcal{A} \cap \mathcal{P} \cap \mathcal{W} \cap \mathcal{E}$
- The last one can be written two ways, either as

$$\mathcal{P} \cap (\mathcal{W} \cup \mathcal{E})$$

or equivalently

$$(\mathcal{P} \cap \mathcal{W}) \cup (\mathcal{P} \cap \mathcal{E})$$

Here are the answers to the question about S_3, S_{11}, S_2 and S_1 from Page 48.

- $S_3 = \{-1, 1\}$.
- $S_{11} = \{-3, 3\}$.
- $S_2 = \{0\}$.
- $S_1 = \{\}$.

The question about when $\mathcal{S} \cap \mathcal{T} = \mathcal{S}$ and when $\mathcal{S} \cup \mathcal{T} = \mathcal{T}$ was given on Page 49.

It turns out that both of those statements will be true if and only if $\mathcal{S} \subseteq \mathcal{T}$. After all, if $\mathcal{S}$ is entirely contained within $\mathcal{T}$, then $\mathcal{S} \cap \mathcal{T}$ is just $\mathcal{S}$. Likewise, if $\mathcal{S}$ is entirely contained within $\mathcal{T}$ then $\mathcal{S} \cup \mathcal{T}$ is just $\mathcal{T}$.

Don't worry at all if you didn't get this one—it is a tough question.