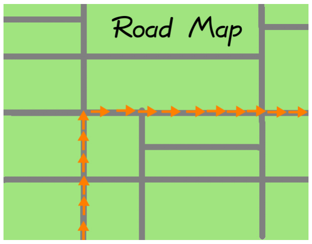
Module 1.5: Intermediate Set Theory and Irrationality



This module will expose you to some intermediate topics of set theory. First, we have some additional concepts of set theory that we need in order to understand the rest of this book. They include the symbols for some very important sets of numbers, set builder notation, power set notation, set subtraction, the Cartesian Product, and exponents of sets.

Second, we will explore the types of numbers (integers, rational, algebraic, real, and complex) with particular focus on determining if numbers are rational versus irrational, and algebraic versus non-algebraic. We'll also see an exact solution of a fourth-degree polynomial, getting formulas that are at once awe-inspiring yet terrifying.

Last, but most important, we'll see a few formal proofs. These are just meant to be read, so that you can become familiar with the language and style of proof writing.

Before we start, I have a warning about notation for you. Many computer engineers use the "overbar" to denote a complement. For example,

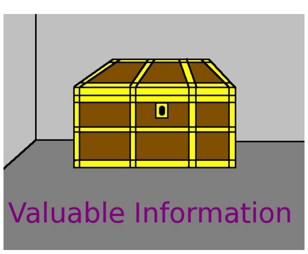
$$\begin{array}{rcl} \mathcal{X}^c & \text{becomes} & \overline{\mathcal{X}} \\ (\mathcal{A}\cap\mathcal{B})^c & \text{becomes} & \overline{\mathcal{A}\cap\mathcal{B}} \\ \mathcal{X}\cap(\mathcal{Y}\cup\mathcal{Z})^c & \text{becomes} & \mathcal{X}\cap\overline{(\mathcal{Y}\cup\mathcal{Z})} \end{array}$$

I have nothing against this notation, and I've used it myself in my own notes when doing research. (Recall, my own bachelor's degree was in Computer Engineering.) Nonetheless, this notation can cause major confusion when grading tests.

Students often will have the overline going over parts of the expression, but in an ambiguous way. In other words, the overline doesn't start and stop in a reasonable position, so it is completely impossible to understand what the student has written. Consider

$$\begin{array}{ccc} \text{horrible notation} \rightarrow & \mathcal{X}\overline{\cap(\mathcal{Y}\cup\mathcal{Z})} & \leftarrow \text{horrible notation} \\ \text{horrible notation} \rightarrow & \mathcal{X}\cap(\mathcal{Y}\overline{\cup\mathcal{Z})} & \leftarrow \text{horrible notation} \end{array}$$

For this reason, I do not use the overline in this book, nor in my research publications. Some textbooks use the overline, but many textbooks avoid it, just as this textbook avoids it. I do not permit students to use this notation on tests when I teach this class. However, if you are taking this class from someone else, your instructor might or might not tolerate it.

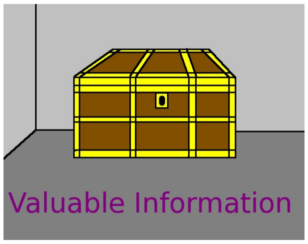


Some sets are so important, and come up so frequently, that they have special symbols. We will discuss them over the next several boxes.

- The set of real numbers is denoted $\mathbb{R}$. That's the set of all the numbers on the number line, just like in your calculus classes.

Note: While ∞ and $-\infty$ are important in calculus, they do not have a location on the number line. Therefore, they are not real numbers.

- The set of *complex numbers* is denoted $\mathbb{C}$, and is the set of all possible sums of the form $a+b\sqrt{-1}$, where a and b are real numbers. This is usually written $a+bi$, where $i=\sqrt{-1}$.



The set of *integers* is denoted

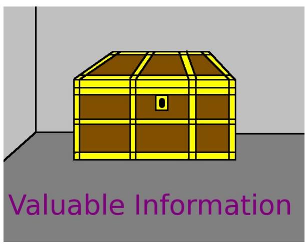
$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

and that set is the heart of number theory.

The choice of the symbol $\mathbb{Z}$ is a bit funny, because the letter "z" does not appear in the word "integer." Why do we use $\mathbb{Z}$ to represent the integers? This is from the German language, where *Zahl* means "number."

Apparently, *Nummer* also means "number," a distinction which I did not understand. Luckily my husband, Patrick Studdard, found an explanation on `wiktionary.org`, which is reproduced below.

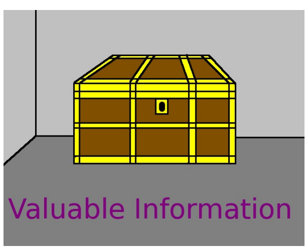
"The German word for an abstract entity used to describe quantity, or a symbol thereof, is *Zahl.* A *Zahl* is only called a *Nummer* when it is used as a means of identification, e.g. in a ranking list, as a phone number, a model number, etc. . . ."



The set of rational numbers is denoted $\mathbb{Q}$. That's the set of all the numbers which can be written as a/b, where a and $b \neq 0$ are both integers. Once again, this seems like kind of an odd choice because the letter "q" does not appear in the word *rational*.

However, you must remember that a fancy word for "the answer to a division" is *quotient*. With that in mind, "$\mathbb{Q}$" isn't so bad of a choice for the symbol of the rational numbers, because every rational number is the answer to infinitely many division problems. For example, $5/3$ is the answer to $5 \div 3$, $10 \div 6$, $15 \div 9$, and so forth.

Another word for a division is a "ratio." That's where the words irrational and rational come from. A number is rational if and only if it can be written as a ratio of two integers. It has nothing to do with rational and irrational lines of thought. When we say $\sqrt{5}$ is *irrational*, we're saying that it cannot be written as a ratio of two integers. We are not saying that $\sqrt{5}$ is criminally insane.

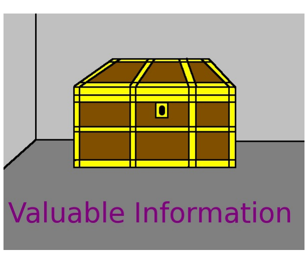


The sets defined above fit together in a very natural way, which you probably have already guessed at. Here are four things to keep in mind.

First, we said that a rational number is a number that could be written as the ratio of two integers a/b, with $b \neq 0$. Second, since $b = 1$ is allowed, we can write $3 = 3/1$ and $-5 = -5/1$, so every integer is a rational number. Thus can say that $\mathbb{Z}$ is a proper subset of $\mathbb{Q}$.

(By the way, if you have forgotten what "proper" subsets are, recall that $\mathcal{S}$ is a proper subset of $\mathcal{T}$ if and only if $\mathcal{S} \neq \mathcal{T}$ and $\mathcal{S} \subseteq \mathcal{T}$. If you need review on this point, please see Page 50 of the module "Introduction to Set Theory.")

We continue our list of four important ideas in the next box.



Third, every rational number can be found on the number line, so $\mathbb{Q}$ is a subset of $\mathbb{R}$. (Actually, the "Theory of Dedekind Cuts" allows us to define real numbers much more deeply and precisely, but it is a bit confusing and abstract, usually covered in more advanced classes. We won't go into detail on that point here.)

Fourth, whenever the square root of an integer is not an integer, it is irrational. Since numbers like $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ are on the number line, but are irrational, then we know $\mathbb{Q}$ is a proper subset of $\mathbb{R}$.

That last concept is a really important point, so please never forget that when the square root of an integer is not an integer, it is irrational. Sadly, we cannot prove that statement just yet, until we develop some more tools. (However, we'll prove that $\sqrt{2}$ is irrational later in this module, on Page 178.)

When I was writing the previous box, I miswrote one of the lines. Instead of

"Whenever the square root of an integer is not an integer, it is irrational."

unfortunately I had written

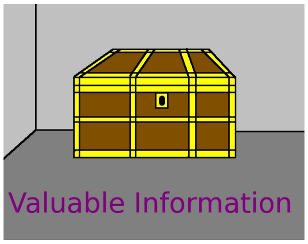
"Whenever the square root of a number is not an integer, it is irrational." $\leftarrow$ WRONG!

which is not correct. The difference is in the seventh word, where I had written "number" in place of "integer." To see why this is wrong, consider

$$\sqrt{\frac{25}{4}} = \frac{\sqrt{25}}{\sqrt{4}} = \frac{5}{2}$$

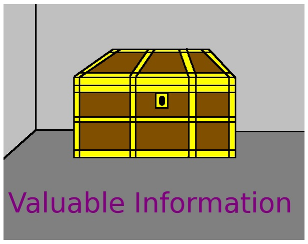
and observe that 5/2 is obviously not irrational—it is the ratio of two integers.

Writing mathematical statements is like defusing a bomb. One tiny mistake, even consisting of a single word, can lead to complete destruction. This reminds me of the story "The Man who was Hanged by a Comma," which we discussed on Page 96 and subsequent pages, of the module "Intermediate Venn Diagram Problems."



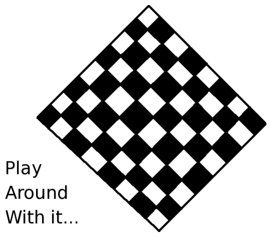
When writing a complex number as $a + b\sqrt{-1}$, or more commonly $a + bi$, we can allow $b = 0$. Therefore, every real number can be found in $\mathbb{C}$. This implies that $\mathbb{R}$ is a subset of $\mathbb{C}$. However, for $a = 0$ and $b = 1$, we have $0 + 1i = i$, and clearly i is not a real number. Therefore, $\mathbb{R}$ is a proper subset of $\mathbb{C}$.

In conclusion, we can write $\mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$ but we will expand this hierarchy later in this module. (Remember, $\subsetneq$ means "proper subset" and $\supsetneq$ means "proper superset.")



When you learned how to work with fractions in elementary school, you learned how to add two fractions, subtract two fractions, multiply two fractions, divide two fractions (taking care not to divide by zero), square fractions, cube fractions, and take the reciprocal of fractions not equal to zero.

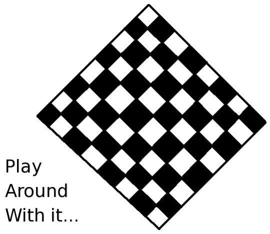
With this in mind, clearly if you have a rational number, and you add, subtract, or multiply by a rational number, it is still rational. If you square or cube a rational number, it is still rational. If you take the reciprocal of a rational number not equal to zero, it is still rational. If you take a rational number and divide by a non-zero rational number, then it is still rational.



1-5-1

For each number, tell me if it is rational or irrational. The answer will be given on Page 183.

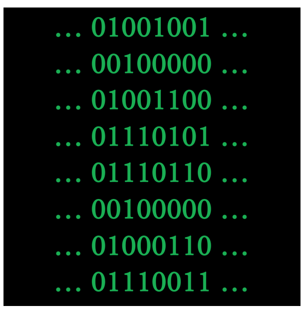
- 17
- 2/3
- $\sqrt{2}$
- 5/4
- $\sqrt{3}/2$
- 1/2
- 0



1-5-2

Once again, for each number, tell me if it is rational or irrational. The answer will be given on Page 183.

- $\sin 90°$, or if you prefer radians, $\sin(\pi/2)$.
- $\sin 60°$, or if you prefer radians, $\sin(\pi/3)$.
- $\sin 45°$, or if you prefer radians, $\sin(\pi/4)$.
- $\sin 30°$, or if you prefer radians, $\sin(\pi/6)$.
- $4^{5/2}$
- $2^{5/2}$
- $\sqrt{20} - 2\sqrt{5}$



There are important consequences to numbers being rational or irrational, when designing mathematical software. I often say that the kingdom of mathematical software can be divided into two phyla. One phylum is exact, and the other is approximate. I'll make this point more clear on Page 192, with an example. Suffice it to say that for now, rational numbers can be represented exactly. Irrational numbers are almost always represented approximately, except in specialized mathematical software.

Toward the end of this module, we're going to see that we can distinguish among irrational numbers. Some are algebraic numbers, and many of those can be represented exactly in computer software, but with great effort. Those that are non-algebraic numbers can only be represented approximately, with a few obscure exceptions.

While the approximations inside of computer software can be rather good, it does lead to rounding error. Moreover, rounding error is like compound interest, in that it starts out small but can snowball to be very large. Generally, if we can avoid it, we should avoid it.

There's a second connection with computer science that I'd like to bring to your attention. If a number is rational, then when written as a decimal, it will either terminate, or eventually repeat a simple pattern over and over again. For example, $1/2 = 0.5$, $1/50 = 0.02$, and $1/8 = 0.125$ are terminating decimals.

Examples of repeating decimals include

$$1/3 = 0.3\overline{3} = 0.333333333\cdots$$

as well as

$$7/198 = 0.035\overline{35} = 0.0353535353\cdots$$

or perhaps

$$1/3996 = 0.000250\overline{250} = 0.000250250250\cdots$$

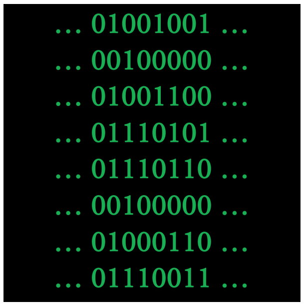
Instead, if a number is irrational, then it will not terminate, and it will not repeat a simple pattern over and over again. Instead, it will proceed to have more and more decimal places. Two famous examples are

$$\sqrt{2} = 1.41421356\cdots$$

as well as

$$\pi = 3.14159265\cdots$$

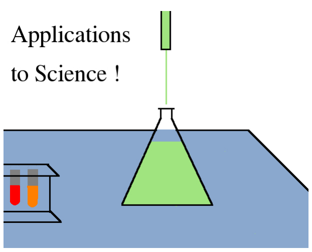
We will continue discussing this point in the next box.



Looking at the irrational numbers in the previous box, we might have some sympathy for high-school teachers who sometimes say that such numbers are "arbitrary," "random," or "have no pattern." All three attributes are completely wrong. There is surely a pattern, and the pattern is most definitely not random—it is entirely deterministic.

The proof of my remark is that there are algorithms for producing those decimal expansions. For example, have you noticed that the newspapers occasionally announce that a computer has computed a new record number of digits of π? Obviously, that computer program has source code, and if read by a programmer, it can be translated into pseudocode and written down as an algorithm.

Because an algorithm produces those digits, the digits are predictable in every sense of the word predictable. That's the exact opposite of random. They are not arbitrary, because if you changed even one of them, you'd have a different number. Somewhat less clear is the fact that if a short computer program can produce millions or billions of digits (given enough time), then surely, there is a pattern.



I have also heard some relatively mathematically savvy people say that these computations of π are useless. One argument is that if you are measuring the area enclosed by the orbit of Pluto, then 33 significant figures is enough to get accurate within 1 sq mm. To be precise,

$$A = \pi(a)b = \pi(5906.38\cdots \times 10^9 \text{ m })(5720.65\cdots \times 10^9 \text{ m }) = 1.06149\cdots \times 10^{26} \text{ m}^2$$

Some explanations are useful. The formula $A = \pi(a)b$ is the area for an ellipse. The a and b take the place of the radius, and are called the semi-major axis and the semi-minor axis. While a is commonly published, b isn't—a fact that I find extremely odd. Yet that's okay, because we can compute b. The eccentricity of an ellipse, ϵ, measures how different it is from a circle, and $b = a\sqrt{1-\epsilon^2}$. The eccentricities of planets, dwarf planets, and asteroids are commonly published, just like their semi-major axes are.

Of course, there are 10^6 sq mm per square meter, therefore

$$A = 1.06149\cdots \times 10^{26} \text{ m}^2 = 1.06149\cdots \times 10^{32} \text{ mm}^2$$

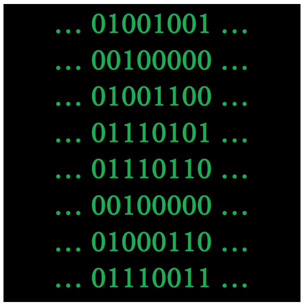
and we can see that 33 significant figures for π is indeed sufficient for this purpose.

While that's undoubtably true, the purpose of calculating millions of digits of π is not for the accuracy of scientific computations. The purpose is very different, and I will reveal it in the next box.

If you are curious, the reason that such enormous numbers of digits of π are computed is so that the accuracy of each generation of new computer can be checked against all previous generations, by seeing if they match or not. In that manner, an error in the arithmetic-logic unit (the ALU) can be detected.

In 1994, a mathematics professor, Thomas Nicely at Lynchburg College in Lynchburg, Virginia, discovered a bug in a then-new microprocessor called the Intel Pentium. Since the pentium was very popular at the time, this created a scandal. The command "`FDIV`" stands for "floating-point division." This command used a lookup table in the microprocessor, to help speed the computation of divisions. Sadly, the lookup table contained an error.

The specific details of how the bug was discovered might be of interest to any computer engineering students who are reading this. They will be given in the next box.



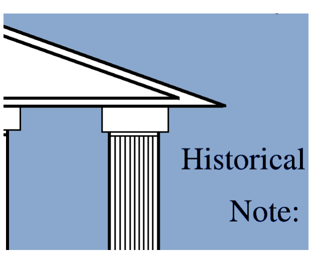
The specific timeline of the events in the previous box will shed light on how serious of an issue this really was. Engineers, and other technology professionals like computer scientists, should always be aware that any mistakes can have extreme consequences.

- On Monday, June 13th, 1994, Prof. Nicely noticed the bug.
- It took until Wednesday, October 19th, 1994, to rule out all other causes.
- Prof. Nicely notified the Intel Corporation on Monday, October 24th, 1994.
- On Sunday, October 30th, 1994, Prof. Nicely asked colleagues on the internet to search for the bug in other microprocessors.
- The first press article about this was an article "Intel fixes a Pentium FPU glitch," by Alexander Wolfe, published in *Electronic Engineering Times*, on November 7th, 1994.
- CNN aired a segment about the bug on Monday, November 21st, 1994.
- The public outcry was so great that on Tuesday, December 20th, 1994, Intel announced that it would replace any flawed Pentium processors upon request.
- By January 1995, the Intel corporation declared for tax purposes that it had lost \$ 475,000,000 replacing those flawed processors, including lost sales and any inventory that had to be discarded. Adjusting for inflation, that is \$ 773,600,000 in July of 2017.

Of course, not every mistake that a computer engineer makes will cost the employer \$ 773,600,000, but it is worth remembering that mistakes have consequences. You can read more about this bug and its discovery in "The Pentium FDIV Flaw FAQ." That FAQ is located at the following URL:

`http://www.trnicely.net/pentbug/pentbug.html`

We will describe the mathematical problem that Prof. Nicely was researching on Page 253 of the module "Fermat's Last Theorem and Some Famous Unsolved Problems."



Earlier, on Page 151, we mentioned the complex numbers $\mathbb{C}$. Some students refuse to listen when being taught about $\mathbb{C}$, and I've heard a few giggles, because of the phrase "imaginary number."

It is a great misfortune of history that $i = \sqrt{-1}$ was given the name "imaginary." You probably have been told that complex numbers have extremely important applications in physics and electrical engineering, and perhaps you have been shown some of them. The problem is that the complex numbers are highly related to the world that we live in, but the word "imaginary" implies the opposite. Numerous mathematics professors and teachers have regretted the unfortunate nickname "imaginary" that complex numbers were somehow given.

In fact, René Descartes (1596–1650), who we will have cause to discuss on Page 165 of this module, used the word "imaginary" in a deliberately derogatory way to discuss these numbers. That was in his work *La Géométrie*, and he considered these numbers imaginary because they could never represent any length, area, or volume in our physical universe. We must forgive Descartes, however, because all of the applications of complex numbers (with the exception of Cardano's work on solving the cubic equation) were found long after Descartes was dead.

You might be interested to know that when solving cubic equations exactly, it is sometimes necessary to go to the complex numbers and come back, even when solving for real roots. This means when you're using Cardano's formula, some of your intermediate steps might be in $\mathbb{C} - \mathbb{R}$ even if the original coefficients, and the final roots, are all in $\mathbb{R}$.

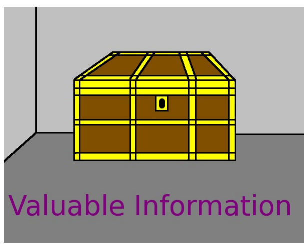
It is unfortunate, but some textbooks use the phrase "the natural numbers" to mean the non-negative integers, $\{0, 1, 2, 3, 4, \ldots\}$, while other textbooks use the phrase "the natural numbers" to mean the positive integers, $\{1, 2, 3, 4, 5, \ldots\}$. Either way, the symbol $\mathbb{N}$ is used to represent the phrase "the natural numbers."

It is a major inconvenience that some textbooks have $0 \in \mathbb{N}$, and some textbooks have $0 \notin \mathbb{N}$. It causes confusion, and can cause a problem on standardized tests. Regardless which convention is being used, we can extend our list from the previous box.

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$$

In recent years, the worldwide community of researchers in both mathematics and in computer science resolved this inconvenience by choosing to explicitly say "the positive integers" or "the non-negative integers" as needed, but never (or rarely) saying "the natural numbers." It is an excellent solution, because now there is no ambiguity at all. Slowly, this resolution of the issue has trickled down into graduate-level textbooks and upper-level undergraduate textbooks.

Accordingly, I will obey this convention. I will say "the positive integers" or "the non-negative integers" as needed, but I will never say "the natural numbers."

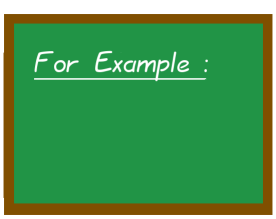


In summary, we can write

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$$

but we will expand upon this hierarchy shortly, concluding with the diagram on Page 174.

I should like to explain set-builder notation now. This notation is extremely popular among pure mathematicians and is likely to appear in any future mathematics courses that you take. Moreover, you've probably seen it before, perhaps without a detailed and precise coverage of the rules and uses of the notation.



1-5-3

First, we're going to translate four examples of set-builder notation into ordinary English, and then we're going to process them further to get the rosters of these four sets.

Our first example:

$$\left\{x \in \mathbb{Q} \,\middle|\, x^2 + x + 1 = 4\right\}$$

means, "the set of all rational numbers x such that $x^2 + x + 1 = 4$."

Our second example:

$$\left\{x \in \mathbb{R} \,\middle|\, x^2 + x + 1 = 4\right\}$$

means, "the set of all real numbers x such that $x^2 + x + 1 = 4$."

Our third example:

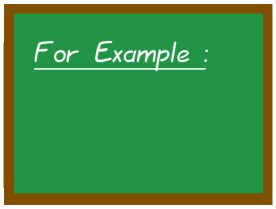
$$\left\{x \in \mathbb{R} \,\middle|\, x^2 + 1 = 0\right\}$$

means, "the set of all real numbers x such that $x^2 + 1 = 0$."

Our fourth example:

$$\left\{x \in \mathbb{C} \,\middle|\, x^2 + 1 = 0\right\}$$

means, "the set of all complex numbers x such that $x^2 + 1 = 0$."



1-5-4

Now let's go back and process those symbols a bit further, so that we can write down the rosters of those sets. We first have to solve the quadratic equation:

$$x^2 + x + 1 = 4$$

We must have "= 0" on the right-hand side, so we subtract 4 from both sides. We obtain

$$x^2 + x - 3 = 0$$

and after blinking at it for a while, we can see that it doesn't factor. So, we roll out the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4(1)(-3)}}{2(1)} = \frac{-1 \pm \sqrt{1+12}}{2} = \frac{-1+\sqrt{13}}{2} \text{ or } \frac{-1-\sqrt{13}}{2}$$

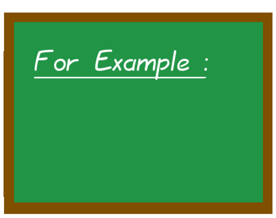
We will continue in the next box.

Continuing with the previous box, now we have to think about how to write down the rosters of our two sets dealing with this quadratic equation. Because both of these roots are clearly irrational,

$$\left\{ x \in \mathbb{Q} \,\middle|\, x^2 + x + 1 = 4 \right\} = \{\}$$

and because both of these roots are clearly real,

$$\left\{ x \in \mathbb{R} \,\middle|\, x^2 + x + 1 = 4 \right\} = \left\{ \frac{-1+\sqrt{13}}{2},\ \frac{-1-\sqrt{13}}{2} \right\}$$



1-5-5

Luckily, the second equation is easy to solve. We are presented with $x^2 + 1 = 0$, and we must first solve that equation. We do so with the following work:

$$\begin{aligned} x^2 + 1 &= 0 \\ x^2 &= -1 \\ x &= \pm\sqrt{-1} \\ x &= \pm i \end{aligned}$$

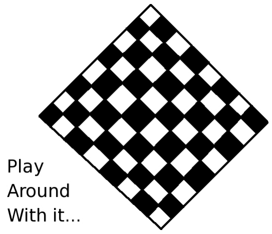
Next, we can say

$$\left\{ x \in \mathbb{R} \,\middle|\, x^2 + 1 = 0 \right\} = \{\}$$

since neither number is on the number line. However, we can also say

$$\left\{ x \in \mathbb{C} \,\middle|\, x^2 + 1 = 0 \right\} = \{-i, i\}$$

There is a technical term for the work that we just did. We have computed the *solution sets* of those two equations, over the last three boxes.

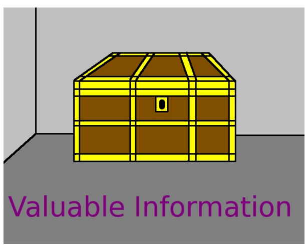


1-5-6

Please practice now by computing the roster of the following solution sets:

- $\left\{x \in \mathbb{R} \,\middle|\, x^3 = x\right\}$
- $\left\{x \in \mathbb{R} \,\middle|\, x^2 + x + 1 = 0\right\}$
- $\left\{x \in \mathbb{C} \,\middle|\, x^2 + x + 1 = 0\right\}$
- $\{x \in \mathbb{Z} \,|\, (x-2)(2x-3) = 0\}$
- $\{x \in \mathbb{Q} \,|\, (x-2)(2x-3) = 0\}$

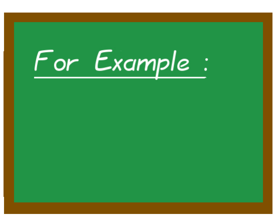
The answers can be found on Page 184.



Sometimes we want to say "the negative reals," "the positive rationals," or "the non-zero complex numbers." There is notation for this as well.

- We write $\mathbb{R}^+$ or $\mathbb{Z}^+$ to say the positive reals or positive integers, respectively.
- We write $\mathbb{R}^-$ or $\mathbb{Z}^-$ to say the negative reals or negative integers, respectively.
- We write $\mathbb{R}^\times$ or $\mathbb{Z}^\times$ to say the non-zero reals or non-zero integers, respectively.

In practice, only the first of those actually gets used often, though you will see the non-zero symbol once in a great while.



1-5-7

Just as the set-builder notation is very useful for working with equations, it is also very useful for working with inequalities. Permit me to show you two examples. First, the symbols

$$\{x \in \mathbb{Z} \,|\, -3 \leq x \leq 3\}$$

indicate all integers x such that $-3 \leq x \leq 3$. The roster of that set is

$$\{-3, -2, -1, 0, 1, 2, 3\}$$

and you'd be surprised how many students forget the 0.

In contrast, the symbols

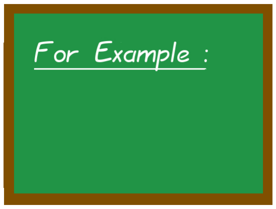
$$\left\{x \in \mathbb{Z}^+ \,\middle|\, -3 \leq x \leq 3\right\}$$

indicate instead

$$\{1, 2, 3\}$$

because we've restricted ourselves to only the positive integers.

An uncommon (but not rare) technical term for what we just did is computing the *truth set* of the given inequalities. We'll see a more interesting example in the next box.



1-5-8

Sometimes the examples are more complicated. Let's suppose that someone asks you to write out the roster of the following set:

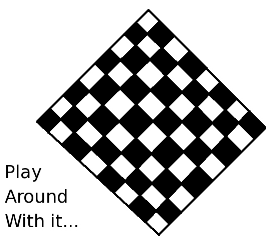
$$\left\{x \in \mathbb{Z}^+ \,|4x - 14 \leq 3\right\}$$

We must first solve the inequality for x. That's not hard at all in this case.

$$\begin{aligned} 4x - 14 &\leq 3 \\ 4x &\leq 17 \\ x &\leq 17/4 \\ x &\leq 4.25 \end{aligned}$$

With that in mind, we can easily identify the positive integers such that $x \leq 4.25$. Clearly, they are

$$\{1, 2, 3, 4\}$$

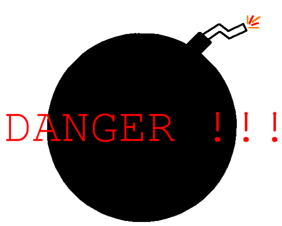


1-5-9

Please practice now by computing the roster of the following truth sets:

- $\{x \in \mathbb{Z} \,|-3 < x < 7\}$
- $\{x \in \mathbb{Z}^+ \,|-3 < x < 7\}$
- $\{x \in \mathbb{Z} \,|-\pi \leq x \leq \pi\}$
- $\{x \in \mathbb{Z}^+ \,|-\pi \leq x \leq \pi\}$
- $\{x \in \mathbb{Z}^+ \,|5x - 19 < 2\}$

The answers can be found on Page 184.



I have to be honest and tell you that when writing

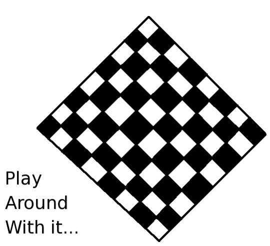
$$\left\{x \in \mathbb{R} \,\middle|\, x^2 + x + 1 = 0\right\}$$

many textbooks will leave out the symbols $\in \mathbb{R}$, writing instead

$$\text{bad notation} \rightarrow \qquad \left\{x \,\middle|\, x^2 + x + 1 = 0\right\} \qquad \leftarrow \text{bad notation}$$

Clearly, this is ambiguous. In one item of the previous box, we wanted the complex roots, because I specifically said $\in \mathbb{C}$. Yet, in another item of the previous box, we didn't want the complex roots, because I specifically said $\in \mathbb{R}$.

As you can see, if you leave out the "$\in \mathbb{R}$" you create ambiguities—therefore no textbook should ever do that.

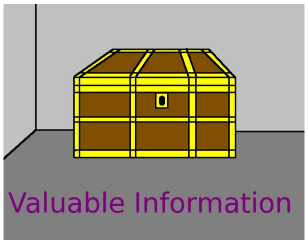


1-5-10

Here is a slightly more challenging problem. Compute the roster of the following set:

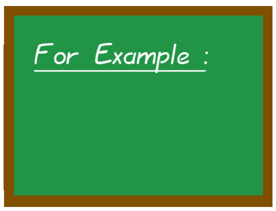
$$\{x \in \mathbb{Z}^+ \,|x > 2\} \cap \left\{x \in \mathbb{R} \,\middle|\, x < \sqrt{61}\right\}$$

The answer will be given on Page 184.



Starting on Page 50 of the module "Introduction to Set Theory," I showed you that a set with n members has 2^n subsets. (This count includes the improper subset and the trivial subset.)

Related to this is the concept of taking *the power set* of a set. The power set of $\mathcal{A}$ is the set of all subsets of $\mathcal{A}$. Sometimes we write it as $P(\mathcal{A})$, but more often we just write "the power set of $\mathcal{A}$."

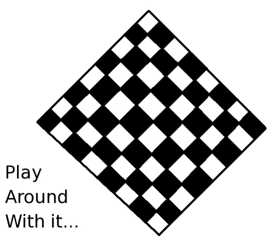


For example, consider the set $\{1, 2, 3, 4\}$. Since we have 4 members, we expect $2^4 = 16$ subsets. The power set is

$$P(\{1,2,3,4\}) = \{\{\};\ \{1\};\ \{2\};\ \{3\};\ \{4\};\ \{1,2\};\ \{1,3\};\ \{1,4\};\ \{2,3\};\ \{2,4\};\ \{3,4\};\ \{1,2,3\};\ \{1,2,4\};\ \{1,3,4\};\ \{2,3,4\};\{1,2,3,4\}\}$$

1-5-11

For an application of the power set concept to medical studies, see Page 52 in the module "Introduction to Set Theory."



1-5-12

Write down the rosters of the following sets:

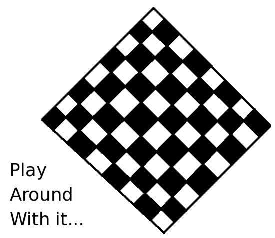
1. What is $P(\{5, 6, 7\})$?
2. What is $P(\{8, 9\})$?
3. What is $P(\{0\})$?
4. What is $P(\{\})$?

The answers will be given on Page 184 of this module.

If you look at the answers to the previous example and the previous checkerboard box, you'll see that indeed...

- ...the set $\{1, 2, 3, 4\}$ had $2^4 = 16$ subsets.
- ...the set $\{5, 6, 7\}$ had $2^3 = 8$ subsets.
- ...the set $\{8, 9\}$ had $2^2 = 4$ subsets.
- ...the set $\{0\}$ had $2^1 = 2$ subsets.
- ...the set $\{\}$ had $2^0 = 1$ subset, namely the empty set.

Clearly, it really does seem to be the case that a set with n members has 2^n possible subsets, at least for $n < 5$. Of course, this does not constitute a mathematical proof!



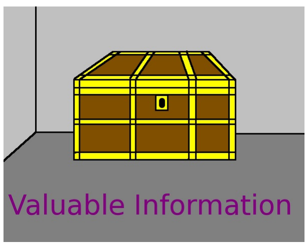
1-5-13

Now I'd like you to replace the "?" in each of the following statements with one of four possible symbols: $\{\subsetneq, \supsetneq, =, \in\}$

(Remember, $\subsetneq$ means "proper subset" and $\supsetneq$ means "proper superset.")

- $\{W, X\}$? $\{W\}$
- $\{W, X\}$? $\{W, X, Y\}$
- $\{\{W, X\}; \{W\}\}$? $P(\{W, X, Y\})$
- $\{W, X\}$? $P(\{W, X, Y\})$

The answers will be given on Page 185.



For two sets $\mathcal{A}$ and $\mathcal{B}$, the notation $\mathcal{A} - \mathcal{B}$ represents the set of things in $\mathcal{A}$ that are not in $\mathcal{B}$. Consider

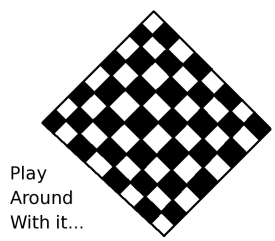
$$\mathcal{X} = \{1, 3, 5, 7\} \text{ and } \mathcal{Y} = \{1, 2, 3, 4\}$$

we would say that

$$\mathcal{X} - \mathcal{Y} = \{5, 7\}$$

By the way, it is worth noting that this operation is called *set subtraction*.

Just so that you're aware, some older books will write $\mathcal{A} \setminus \mathcal{B}$ instead of $\mathcal{A} - \mathcal{B}$, but that notation is becoming rare.



1-5-14

Consider

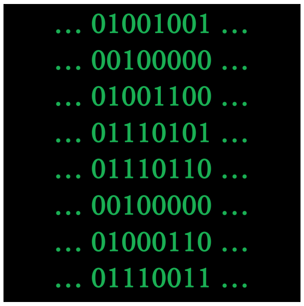
$$\mathcal{X} = \{1, 3, 5, 7\} \text{ and } \mathcal{Y} = \{1, 2, 3, 4\} \text{ as well as } \mathcal{Z} = \{4, 5, 6, 7\}$$

and with that definition in mind, write down the rosters of the following sets:

1. What is $\mathcal{X} - \mathcal{Z}$?
2. What is $\mathcal{Y} - \mathcal{X}$?
3. What is $\mathcal{Y} - \mathcal{Z}$?
4. What is $\mathcal{Z} - \mathcal{X}$?
5. What is $\mathcal{Z} - \mathcal{Y}$?
6. What is $\mathcal{Z} - \mathcal{Z}$?

The answer will be given on Page 185.

At first glance, it is possible that some readers might imagine that the set subtraction operation is just a toy of pure mathematics, with no applications. Actually, the exact opposite is true. In general, almost all sorts of databases will want to have some way of expressing the "without" operation. In particular, I'd like to share with you one of my favorite uses of this operator, which is the minus operator for Google searches.



In the year 2003, a particular book of fiction was a favorite of the general mathematics/computer science/engineering community. I am speaking of Dan Brown's *The DaVinci Code*. Later, in the year 2006, it was made into a movie.

Let's suppose that I want to use a quote, which I half-remember from that book, while preparing a lecture for my discrete mathematics class. Alternatively, suppose that I want to collect a few reviews of the book to help convince some other faculty member that the book is worth talking about. Naturally, many more references on the internet will be talking about the movie than will be talking about the book. The number of people who watch movies in the world is much larger than the number of people who read nerdy books.

Therefore, if I were to type the following into Google

```
davinci code
```

then most of the top hits would be about the movie, and not the book. Instead, if I type

```
davinci code -movie
```

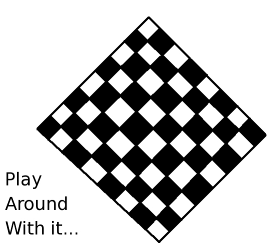
then Google knows that I want the set of webpages that contain "`davinci`" and "`code`" but subtracting out the set of webpages that contain "`movie`." (There is a minus sign before the word movie, in case you didn't notice it.) I will be much better served by this query, because I will only get webpages that talk about the book.

I can imagine a student challenging me on this point. Do we really need set subtraction for the above? Can't we just use the following:

$$\texttt{davinci} \cap \texttt{code} \cap \texttt{movie}^c$$

While that would be logically equivalent (mathematically), it would be a disaster in terms of how databases actually work in the real world. That's because the set $\texttt{movie}^c$ is the set of all webpages on the internet which do not contain the word movie. That set is phenomenally huge! The vast majority of the webpages on the internet do not contain the word movie! Computing $\texttt{movie}^c$, in order to compute the intersection, would cause a server to run out of memory and crash.

When working with or programming databases, it is crucial to keep such issues in mind. Otherwise, your queries will take far too long to process (minutes instead of milliseconds) and that might be extremely frustrating for your users.



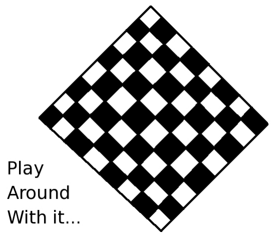
1-5-15

I'd like to ask you a more general question about set subtraction. Can you tell me (in general) whether

$\mathcal{A} - \mathcal{B} = \mathcal{B} - \mathcal{A}$ is true or false for all sets?

Hint: look at your answers for $\mathcal{X} - \mathcal{Z}$ and $\mathcal{Z} - \mathcal{X}$ from the previous question.

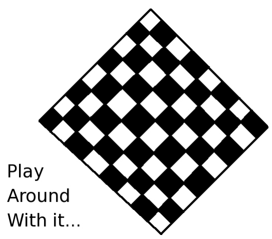
The answer will be given on Page 185 of this module.



1-5-16

Think back to Module 1.3: "Intermediate Venn Diagram Problems," where we shaded 2-circle Venn Diagrams with yellow and white, in order to show what various set-theoretic formulas meant. In that style, draw a Venn Diagram for $\mathcal{A} - \mathcal{B}$.

The answer will be given on Page 185.

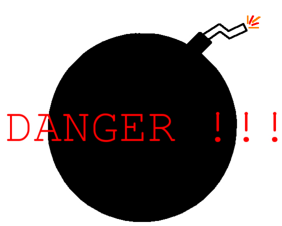


1-5-17

Here are some formulas from set theory. Replace the "?" in each formula with one of the four following symbols: $\oplus, \cap, \cup, -$

- $\{A, B, C, D\} \ ? \ \{A, C, E\} = \{B, D, E\}$
- $\{A, C, E\} \ ? \ \{B, C\} = \{A, E\}$
- $\{A, B, C, D, E\} \ ? \ \{\} = \{\}$
- $\{A, C, E\} \ ? \ \{C\} = \{C\}$
- $\{A, B, C, D\} \ ? \ \{A, C, E\} = \{B, D\}$
- $\{A, C, E\} \ ? \ \{C\} = \{A, C, E\}$
- $\{A, B, C, D\} \ ? \ \{A, C, E\} = \{A, B, C, D, E\}$
- $\{A, C, E\} \ ? \ \{B, C\} = \{A, B, E\}$
- $\{A, B, C, D\} \ ? \ \{A, C, E\} = \{A, C\}$

The answers will be given on Page 186.



In an earlier draft of this module, I had made an error while writing the previous question. That error is worth dwelling upon.

I had written the following question:

$$\{A, C, E\} \ ? \ \{C\} = \{A, E\}$$

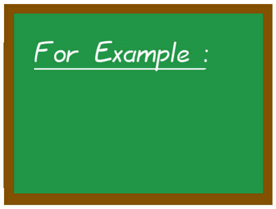
However, as you can see, this can be answered either this way:

$$\{A, C, E\} \oplus \{C\} = \{A, E\}$$

or equivalently it can be answered this way:

$$\{A, C, E\} - \{C\} = \{A, E\}$$

Even though set theory seems completely trivial, there are a lot of subtle points in set theory.



1-5-18

There is a minor but useful bit of notation called the *Cartesian Product*. When we write $\mathcal{A} \times \mathcal{B}$, we mean all possible ordered pairs, with the first member coming from $\mathcal{A}$ and the second member coming from $\mathcal{B}$. For example

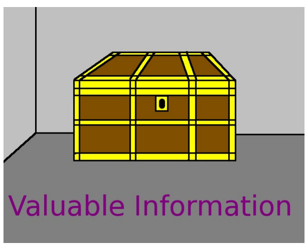
$$\{1,2\} \times \{3,4,5\} = \{(1,3);\ (1,4);\ (1,5);\ (2,3);\ (2,4);\ (2,5)\}$$

and similarly

$$\{x,y,z\} \times \{a,b\} = \{(x,a);\ (x,b);\ (y,a);\ (y,b);\ (z,a);\ (z,b)\}$$

Did you notice that we prohibit $(1,2)$ in the first example, and (x,y) in the second example? That's because $2 \notin \{3,4,5\}$ and $y \notin \{a,b\}$.

Moreover, $(3,1)$ is completely out of the question in the first example, because $3 \notin \{1,2\}$ and $1 \notin \{3,4,5\}$. This remains true even though $(1,3)$ is a member of the first example.



In the previous box, we saw that $(1,3)$ was a member of $\{1,2\} \times \{3,4,5\}$ but that $(3,1)$ was not a member of $\{1,2\} \times \{3,4,5\}$. This reveals the real distinction between an ordered pair and a set. With ordered pairs

$$(1,3) \neq (3,1)$$

whereas with sets

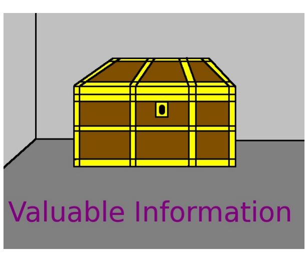
$$\{1,3\} = \{3,1\}$$

because order does not matter inside a set.

If you keep this distinction in mind, then you'll never forget when to use parentheses, e.g. $(1,3)$, versus braces, e.g. $\{1,3\}$. The parentheses mean that order matters, while the curly braces mean that order doesn't matter. That's a very important point!

And while mathematical notation can be very strict at times, it is amazingly powerful. There's a lot of depth involved, and the symbols carry a lot of hidden meaning. In this case, the choice of parentheses versus braces tells you whether order matters or order does not matter.

Perhaps you understand now why professors enforce notation strictly, even if you wish that they would not.



While the above notational distinction is important (showing whether or not order matters, by using either parentheses or braces), the following notational distinction is merely a convenience. It applies to situations like the power set or the Cartesian product, where we have a set of sets or a set of ordered pairs.

The small sets (or ordered pairs) making up the larger set will have commas separating the entries. The set of sets (or set of ordered pairs) will have semicolons separating the small sets (or small ordered-pairs) from each other.

Though this might seem like a tedious distinction, it really helps. It is also good etiquette to put more of a space after each semicolon than after each comma.

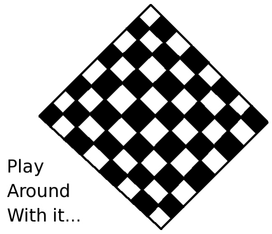
To illustrate the effectiveness of this convention, consider

$$\{x,y,z\} \times \{a,b\} = \{(x,a),(x,b),(y,a),(y,b),(z,a),(z,b)\} \leftarrow \text{ imperfect notation}$$

in contrast with

$$\{x,y,z\} \times \{a,b\} = \{(x,a);\ (x,b);\ (y,a);\ (y,b);\ (z,a);\ (z,b)\} \leftarrow \text{ better notation}$$

and think about which is easier to read.

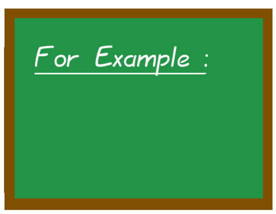


1-5-19

Practice your mastery of the Cartesian Product by computing the following:

1. $\{0, 3\} \times \{1, 2, 4, 8\}$
2. $\{1, 2, 3, 4, 5\} \times \{0\}$
3. $\{a, b, c\} \times \{x, y, z\}$
4. $\{a, b\} \times \{b, c\}$

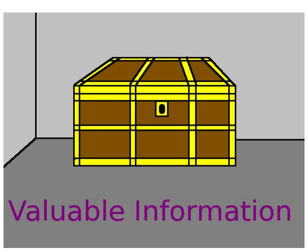
The answers will be given on Page 186.



1-5-20

You might or might not have noticed a pattern in the sizes of those Cartesian products.

- $\{x, y, z\} \times \{a, b\}$ had 6 members, while $\#\{x, y, z\} = 3$, and $\#\{a, b\} = 2$.
- $\{1, 2, 3, 4, 5\} \times \{0\}$ had 5 members, while $\#\{1, 2, 3, 4, 5\} = 5$, and $\#\{0\} = 1$.
- $\{0, 3\} \times \{1, 2, 4, 8\}$ had 8 members, while $\#\{0, 3\} = 2$, and $\#\{1, 2, 4, 8\} = 4$.
- $\{a, b, c\} \times \{x, y, z\}$ had 9 members, while $\#\{a, b, c\} = 3$, and $\#\{x, y, z\} = 3$.
- $\{a, b\} \times \{b, c\}$ had 4 members, while $\#\{a, b\} = 2$, and $\#\{b, c\} = 2$.

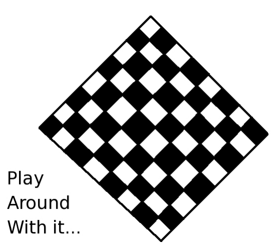


Looking at the data in the previous box, we can presume that

$$\#(\mathcal{A} \times \mathcal{B}) = (\#\mathcal{A})(\#\mathcal{B})$$

which is just mathematical notation for "the size of the Cartesian product is the product of the sizes of the sets." This is not, however, a proof.

With that formula in mind, it seems reasonable that the symbol chosen for the Cartesian product was $\times$.



1-5-21

Before we continue discussing the Cartesian Product of two sets, I'd like to challenge you to find a formula for me. We just learned that

$$\#(\mathcal{A} \times \mathcal{B}) = (\#\mathcal{A})(\#\mathcal{B})$$

which is mathematical notation for "the size of the Cartesian product is the product of the sizes of the sets." Can you produce a similar formula, for size, during set subtraction?

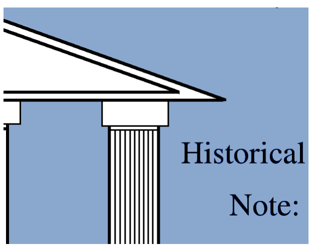
$$\#(\mathcal{A} - \mathcal{B}) = ???$$

Take a few minutes to think about this. Be sure to test your hypothesis with a few examples before peeking at the answer, which will be found on Page 186. This question is somewhat difficult, but try to find the answer anyway.

You might or might not remember that the ordinary coordinate plane (as compared to polar coordinates, curvilinear coordinates, cylindrical coordinates, or spherical coordinates) is attributed to René Descartes (1596–1650). (Pronounced RAY-nay-day-CART.) That's why the ordinary coordinate plane is called the Cartesian coordinate system.

You might wonder if it is the same Descartes when we talk about the Cartesian Product. Indeed, it is the same person. He introduced the coordinate plane, bridging algebra and geometry, and also wrote the essay about the evil genius (predicting virtual reality in the 1630s, roughly 3.5 centuries before it became available), as well as coming up with the famous quote "Je pense, donc je suis!" or "Cogito ergo sum!" Regrettably, because the younger generation (at least in the USA) is highly allergic to foreign languages, this is now often taught as "I think, therefore I am!" or more correctly, "I think, therefore I exist!"

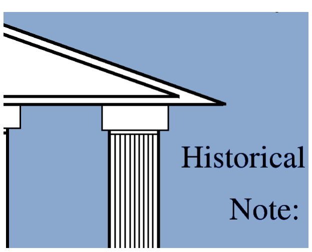
While that quote is often repeated, it is not as important as other contributions by Descartes, which we will discuss in the next box.



Descartes's essay "A Discourse on Method" reintroduced the scientific method to European thought, emphasizing the need to question everything and take nothing on faith. This attitude entered European thought with the ancient Greeks, primarily through Aristotle (384–322 BCE) and his numerous debating partners, but was lost in Western Europe with the Fall of the Western Roman Empire in 476 CE. The reason for the complete loss of the scientific perspective should be no mystery to any educated person, because it is rather simple.

It is indisputable that in the classical period, scientific work, even five hundred years after Aristotle was dead, was invariably written in Greek. Take as your example Diophantus of Alexandria (208±7 – 292±7 CE) who wrote in Greek even though he lived in Egypt.

After the fall of the Western Roman Empire, very few people in Western Europe could read Latin, and far fewer studied Greek, so they could not read these writings at all. How can we expect them to have known about things explained in books which they could not possibly read?



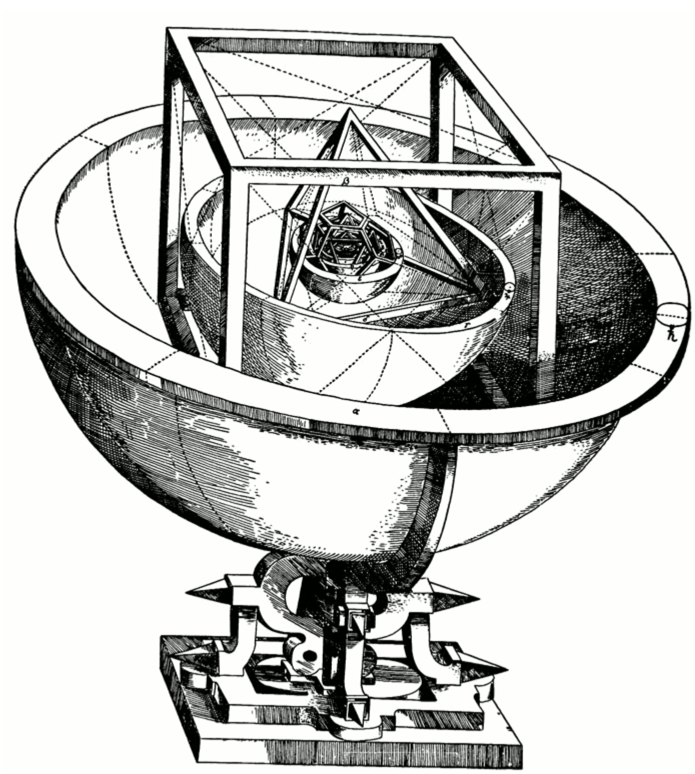
Most significantly to anyone who wears eye glasses or contact lenses, Reneé Descartes wrote an important essay *Dioptrics* about optics, introducing what we typically call Snell's Law and the index of refraction. In fact, many believe that he wrote *Dioptrics* to give an example of what the scientific method could accomplish. Snell's Law had been independently discovered by Thomas Harriot in 1602, and Willebrord Snellius in 1621, but neither of them published their work. Furthermore, the scientist Abu Sa'd al-'Ala' ibn-Sahl (940±10–1000), working in Baghdad but of Iranian ancestry, published it in 984, yet it remained unknown in Europe because the Islamic and Christian communities were not on speaking terms during those particular centuries. (This is, perhaps, a grave understatement, considering the extreme violence of The Crusades.) Descartes has also given us the Law of the Conservation of Momentum, in the one-dimensional case.

In any case, he was a busy and prolific writer. How could such an amazing genius come about? Well, if you look closely, you will see that his lifespan almost perfectly aligns with the Golden Age of French thought. That era began with the Edict of Nantes in 1598 (when Descartes was two years old) and ended with the Edict of Fontainebleau in 1685, after Descartes had passed away. Sometimes this Golden Age is called "the 87 years." We will discuss those years further on Page 231 and Page 256 of the module "Fermat's Last Theorem and Some Famous Unsolved Problems" and also on Page 361, in the module "The Square-Root of NPQ Rule."

Even though he lived during the 87 years, Renee Descartes made a great discovery that he kept hidden from the world at large. He proved a formula for the convex polyhedra that is known today as Euler's formula: $F - E + V - 2 = 0$. (Actually, several other formulas are also called Euler's formula as well.) We will discuss this formula later in the course.

Descartes also proved that the five known platonic solids—the icosahedron, the dodecahedron, the octahedron, the cube, and the tetrahedron, known to those who have played Dungeons & Dragons as the d20, the d12, the d8, the d6, and the d4—are the only possible regular convex polyhedra.

These are major results, but the fact of the matter is that in Plato's work *Timaeus*, and in later writings by Aristotle, those five polyhedra were the shapes of the atoms of the five elements thought to exist at the time (water, aether, air, earth, and fire) and therefore those shapes were associated with alchemy. Descartes was worried that non-mathematicians would misunderstand his formula and his proof, and imagine it to be about alchemy and not about math. This could result in Descartes being executed.



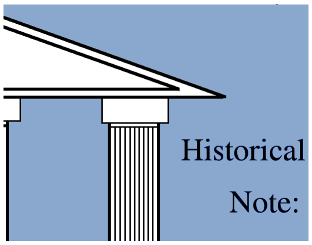
An additional discovery refers to a relationship between the distances from the sun to the first few planets and the ratios of the sizes of those polyhedra when placed one-inside-the-other. You start with a sphere at the center, then an octahedron, then a sphere, then an icosahedron, another sphere, a dodecahedron, yet another sphere, a tetrahedron, still another sphere, a cube, and finally one last sphere. Each object is made as small as could be possible given the arrangement. The spheres have, as great circles, the orbits of Mercury, Venus, Earth, Mars, Jupiter, and finally Saturn. The point at the center of the whole thing is the sun—making this a heliocentric model, which was yet another reason that Descartes notebook could have gotten him executed.

Johannes Kepler (1571–1630) had come up with this beautiful hypothetical relationship. Johannes Kepler published it in 1596 in his book *Mysterium Cosmographicum*, and Descartes probably knew about it. This means that an outsider, seeing the platonic solids, would think that Descartes' notebook was about astronomy, and therefore astrology. This is a third reason that Descartes' secret notebook could have gotten him executed.

Not too many years later, the data from astronomical observations proved this model false, which Kepler freely admitted during his own lifetime. In fact, Kepler came up with his laws of planetary motion as a "second try" after his first model failed. Among other things, Kepler figured out that the orbits aren't circles at all (though that's a good approximation). The orbits of the planets are ellipses!

Here are lessons for all of us. When we are proven wrong, we should admit it gracefully. When we experience temporary setbacks, we should learn from them and use that knowledge to produce eventual successes.

The drawing in the previous box was made in the year 1596, or earlier, and is not subject to copyright in the year 2017. Nonetheless, I offer this academic citation to Johannes Kepler, and to Wikimedia Commons for providing a very nice scan of the drawing.



For the reasons explained in the previous two boxes, Renee Descartes kept the notebook hidden. It is known that Gottfried Wilhelm Leibniz (1646–1716), the co-inventor of calculus, copied part of the notebook after Descartes had died. Even though the notepad was enciphered, Leibniz broke the code. Interestingly, Leibniz did not publish the formula $F - E + V - 2 = 0$, either. We don't know if it was out of fear, or because he was not the discoverer.

This formula became known to the world when Euler published in it in the year 1752, sadly 102 years after Descartes had died. You can read more about the story of this notebook in *Descartes' Secret Notebook: A True Tale of Mathematics, Mysticism, and the Quest to Understand the Universe* by Amir Aczel, published by Broadway Books in 2006.

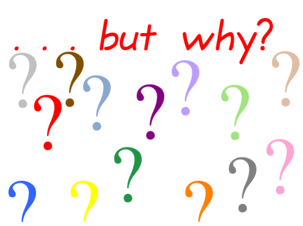
At this point, some students might assume that because the Cartesian product looks useless, that it is unrelated to computer science. They might imagine that this is merely some sort of pure mathematics exercise, forming a type of Calisthenics of the Mind.

Of course, a more thoughtful student would understand that I would not be including it in the textbook at all if it were not connected to applications. Let's discuss those applications now.

The Cartesian Product is part of a hugely important operation called "a cross join," and it is used in relational databases.

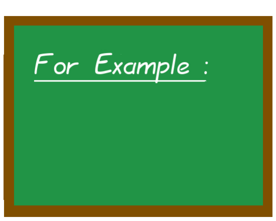
Suppose you have a database with a set of instructors, $\mathcal{I}$, and a set of classes $\mathcal{C}$. The course assignments for the coming term are just a subset of $\mathcal{I} \times \mathcal{C}$, in the sense that each course assignment is an ordered pair, connecting an instructor to a course.

Likewise, an online retailer has lots of products, and most might be at one or two of many warehouses, while some might be at most warehouses. The database that tracks this will have a set for products, $\mathcal{P}$, and a set of warehouses $\mathcal{W}$. A subset of $\mathcal{P} \times \mathcal{W}$ will tell you which products are at which warehouses.



The previous box discusses the secondary use of the Cartesian Product. The primary use has to do with the concept of an equivalence relation. It is a cornerstone of Discrete Mathematics, especially as it connects with other advanced topics in mathematics.

We will discuss equivalence relations much later. Yet, rest assured that it depends heavily upon the idea of the Cartesian product.



Suppose you have the set

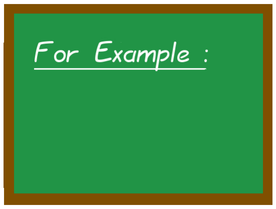
$$\mathcal{S} = \{4, 5, 6\}$$

and someone asks you to write the roster of $\mathcal{S}^2$.

When we work with numbers, we know that $7^2 = (7)(7) = 49$ and therefore it makes sense that

$$\mathcal{S}^2 = \mathcal{S} \times \mathcal{S} = \{(4,4);\ (4,5);\ (4,6);\ (5,4);\ (5,5);\ (5,6);\ (6,4);\ (6,5);\ (6,6)\}$$

1-5-22



Similar to the previous box, when we work with numbers, we know that $8^3 = (8)(8)(8)$. Consider

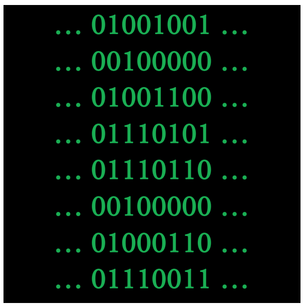
$$\mathcal{T} = \{7, 8\}$$

and suppose that someone asks you to write the roster of $\mathcal{T}^3$.

I hope it makes sense that

$$\mathcal{T}^3 = \mathcal{T} \times \mathcal{T} \times \mathcal{T} = \{(7,7,7);\ (7,7,8);\ (7,8,7);\ (7,8,8);\ (8,7,7);\ (8,7,8);\ (8,8,7);\ (8,8,8)\}$$

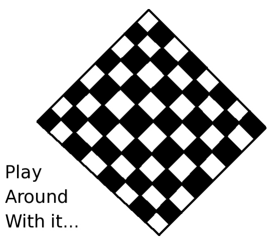
1-5-23



It is in social media that the operation of squaring a set or cubing a set is most relevant. Let $\mathcal{U}$ be the set of users on Facebook. The data of who is friends with whom is just a subset of $\mathcal{U}^2$, because it is a set of ordered pairs—each entry being from $\mathcal{U}$.

Most of the time, friendships are reciprocal. However, on Twitter, it is possible for me to be a follower of some famous person, and probably that famous person is not following me. This is an example of a one-way friendship, and that's why we need to represent friendship with ordered pairs and not sets of size two.

Sometimes if one of my Facebook friends has a friend who is not one of my friends, Facebook will suggest that I add that person to my list of friends. In that case, we are looking for ordered triples (a, b, c) such that a is friends with b and b is friends with c, but a is not friends with c. The set of ordered triples to be considered is clearly a subset of $\mathcal{U}^3$.



- Consider $\mathcal{C} = \{1, 0, -1\}$. What is $\mathcal{C}^2$?
- Consider $\mathcal{D} = \{0, 1\}$. What is $\mathcal{D}^3$?

The answers will be given on Page 187.

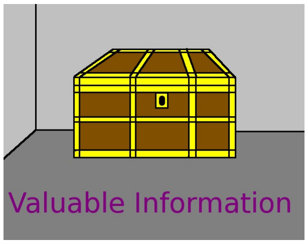
1-5-24

In multivariate calculus, sometimes called *Calculus iii*, we sometimes see the notation $\mathbb{R}^2$ and $\mathbb{R}^3$. Now, you know what it means.

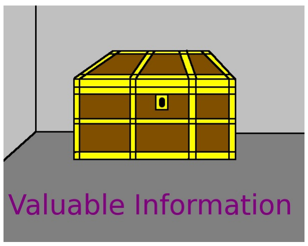
- By $\mathbb{R}^2$, we mean the set of all possible ordered pairs, where each entry is from $\mathbb{R}$. That's the ordinary coordinate plane, formally called the Cartesian coordinates.
- By $\mathbb{R}^3$, we mean the set of all possible ordered triples, where each entry is from $\mathbb{R}$. That's ordinary three-dimensional space. You've been living in it for your entire life.

As life, technology, and mathematics evolve, it comes to pass that some topics become obsolete. However, sometimes new topics move from fourth-class or third-class level to second-class topics. Normally, the concept of an *algebraic number* is not included in a Discrete Mathematics or Discrete Structures course.

Yet, it is very important in computer algebra and as computer algebra systems do more and more of our mathematics for us, topics like these are growing in importance. Whether a number is algebraic or not, and of what degree, is crucial in determining if it can be expressed exactly. Finally, the concept of an algebraic number is highly relevant in pure mathematics.



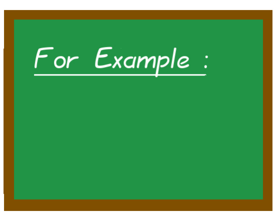
Recall that we say a number r is a *root of a polynomial* $f(x)$ if and only if $f(r) = 0$. In other words, the set of roots of a polynomial function is the set of inputs that make the output of the function equal to zero.



Another nice distinction among numbers is that of an *algebraic number*. Any number that is the root of a polynomial with integer coefficients is an algebraic number. We can write $\mathbb{A}$ to represent the set of algebraic numbers.

(Actually, a more popular symbol for the set of algebraic numbers in textbooks is $\overline{\mathbb{Q}}$, but to me, that looks too similar to $\mathbb{Q}$. So in this textbook, we'll use $\mathbb{A}$, which is the second-most common symbol.)

The next two boxes will show you some key tools for determining whether a number is algebraic.



\# 1-5-25

For example, $\sqrt{2}$ is a root of $x^2 - 2 = 0$, and $\sqrt{5}$ is a root of $x^2 - 5 = 0$. Accordingly, for any positive integers a and b, we know that

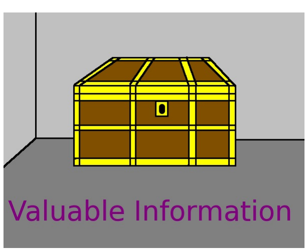
$$\sqrt[a]{b} \text{ is a root of the polynomial } x^a - b = 0$$

because of the simple fact that

$$\left(\sqrt[a]{b}\right)^a - b = b - b = 0$$

which means that $\sqrt[a]{b}$ is algebraic.

Similarly, you can use $x^a + b = 0$ to make the same argument for $-\sqrt[a]{b}$ being algebraic.



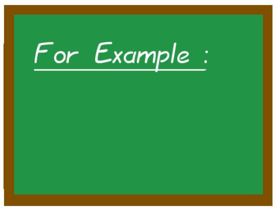
Likewise, any rational number a/b is algebraic because it is a root of the polynomial $bx-a = 0$. In this case, we should allow a to be any integer, and b can be any non-zero integer. To see why a/b is algebraic, observe

$$b(a/b) - a = a - a = 0$$

therefore a/b is a root of the polynomial $bx - a = 0$.

In terms of set theory, this means that $\mathbb{Q} \subseteq \mathbb{A}$, because any number in $\mathbb{Q}$ can be written as a/b for two integers a and b, with $b \neq 0$.

Though it is extraordinarily difficult to prove, both π and e are non-algebraic.



1-5-26

Let's try to figure out whether or not the number

$$\frac{13+\sqrt{47}}{19}$$

is an algebraic number.

At first, this looks daunting, until we realize that this number looks like it emerged from the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

We can see that $-b = 13$, and $2a = 19$. Of course, that implies that $b = -13$ and $a = 19/2$. Now we know that $b^2 - 4ac = 47$, so we just solve for c. We have the following steps:

$$\begin{aligned} b^2 - 4ac &= 47 \\ (-13)^2 - 4(19/2)c &= 47 \\ 169 - 38c &= 47 \\ -38c &= 47 - 169 \\ -38c &= -122 \\ c &= -122/(-38) \\ c &= 61/19 \end{aligned}$$

At this point, we have the polynomial

$$0 = ax^2 + bx + c = \frac{19}{2}x^2 - 13x + \frac{61}{19}$$

and we can clear denominators by multiplying by 38, obtaining

$$0 = 361x^2 - 494x + 122$$

I'd like to share two warnings about the previous problem. First, you have to work with fractions—not decimals—in this case. Otherwise, it would be very hard to end up with an all-integer polynomial. We really do need to know what the denominators are in order to clear denominators. It would be tough to just recognize, from a decimal, that

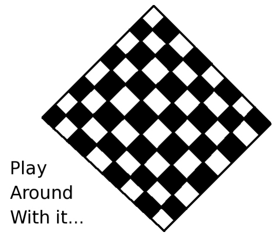
$$3.21052631\cdots = 61/19$$

Second, if you are working with a friend, and your friend gets an integer multiple of this polynomial, then you should both understand that you have "the same answer." For example, if I write down the above polynomial times 2, I would obtain

$$0 = 722x^2 - 988x + 244$$

and similarly if I multiplied by -1 I would obtain

$$0 = -361x^2 + 494x - 122$$

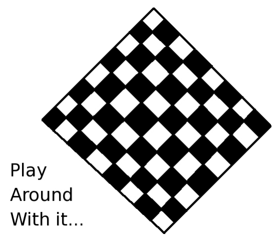


1-5-27

For each of these numbers, tell me if the number is algebraic or not. If so, then give me any polynomial (with integer coefficients) for which it is a root.

- $\sqrt[19]{101}$
- $\frac{1+\sqrt{5}}{2}$
- $1/\sqrt{5}$
- $\sqrt{3}/2$
- $\sin 60°$

The answers will be given on Page 187.

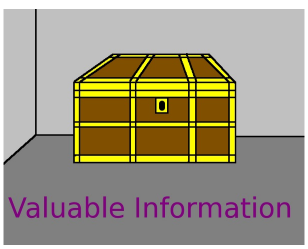


1-5-28

For what positive integers z is the number

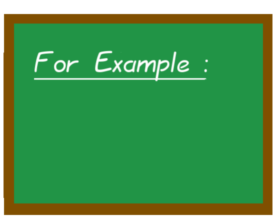
$$\sqrt[z]{2}$$

an algebraic number?



The *degree* of an algebraic number r is the degree of the lowest degree polynomial $p(x)$ such that $p(r) = 0$.

In other words, we must ask ourselves "What is the lowest degree polynomial $p(x)$ such that $p(x)$ has r as a root?" That polynomial is called a *minimal polynomial* for r.



1-5-29

While we know that the numbers

$$\sqrt{4},\quad \sqrt[3]{4},\quad \sqrt[4]{4},\quad \sqrt[5]{4},\quad \sqrt[6]{4},\quad \sqrt[7]{4},\quad \sqrt[8]{4},\quad \sqrt[9]{4},\quad ,\ldots$$

are algebraic because they are the roots of the polynomials

$$x^2-4=0,\quad x^3-4=0,\quad x^4-4=0,\quad x^5-4=0,\quad x^6-4=0,\quad x^7-4=0,\quad x^8-4=0,\quad x^9-4=0,\quad \ldots$$

It turns out that their degrees are a little more complicated, being

$$1, 3, 2, 5, 3, 7, 4, 9, \ldots$$

which is a bit surprising.

For example, while it is true that $x^4 - 4 = 0$ has $\sqrt[4]{4}$ as a root, it is also true that

$$\sqrt[4]{4} = \sqrt{\sqrt{4}} = \sqrt{2}$$

and therefore $\sqrt[4]{4}$ is a root of $x^2 - 2 = 0$. We know $\sqrt{2}$ is irrational, so no first-degree polynomial will work. Thus, $\sqrt[4]{4}$ is an algebraic number of second degree.

An algebraic number of degree one is always a rational number. Can you explain why? As an aside, it is for this reason that I said "No first-degree polynomial will work" in the previous box.

Suppose a non-zero algebraic number r is of degree one. That means r is the root of some polynomial of degree 1. Call that polynomial $ax + b = 0$, recalling that a and b are integers, with $a \neq 0$. Then we have

$$\begin{aligned} ar + b &= 0 \\ ar &= -b \\ r &= -b/a \end{aligned}$$

Since $r = -b/a$, we have written r as a quotient of two integers, which is the very definition of a rational number. Therefore, all algebraic numbers of degree one are rational.

You might be wondering why I said $a \neq 0$ with such confidence, in the previous box. Note that we do not consider the polynomial

$$0x^3 + 2x^2 - 3x + 5 = 0$$

to be cubic, but instead we consider it to be quadratic. The degree of a polynomial is the highest exponent on any term with a non-zero coefficient.

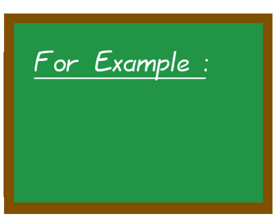
This is a necessary idea. Because of the fact that

$$2x^2 - 3x + 5 = 0x^3 + 2x^2 - 3x + 5 = 0x^4 + 0x^3 + 2x^2 - 3x + 5 = 0x^5 + 0x^4 + 0x^3 + 2x^2 - 3x + 5$$

we could never have any concept of degree unless we ignored all terms with the coefficient equal to zero, drawn here in red ink. The terms in blue ink are the ones that matter.

Consequently, the polynomial $ar + b = 0$ would not be degree one if $a = 0$. Instead, it would be a degree zero polynomial, which is usually called a constant polynomial. (For example, $f(x) = 7$ is a constant polynomial.) Since our problem dealt with algebraic numbers of degree one, then $a \neq 0$.

You will often find that the flaw in any proof with only a minor flaw is related to the number zero in some way. On the other hand, major flaws are more varied.

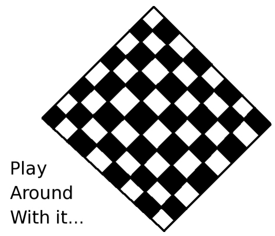


Let's consider now the algebraic degrees of $\sqrt{4}$ and $\sqrt[3]{4}$.

Clearly $\sqrt{4} = 2$, and that is rational. Using the argument from two boxes ago, we can see that $\sqrt{4} = 2$ is a root of the polynomial $x - 2 = 0$. Thus, $\sqrt{4}$ is an algebraic number of the first degree.

For $\sqrt[3]{4}$, we can write $x^3 - 4 = 0$, and see that $\sqrt[3]{4}$ is an algebraic number of the third degree.

1-5-30



1-5-31

Give me polynomials to justify the following statements. Hint: $4 = 2^2$.

- The algebraic number $\sqrt[6]{4}$ is of the third degree.
- The algebraic number $\sqrt[8]{4}$ is of the fourth degree.
- The algebraic number $\sqrt[6]{27}$ is of the the second degree.
- The algebraic number 5 is of the first degree.

The answer can be found on Page 187.

As you can see, the concept of an algebraic number is a concise and easily understood notion. Then you might be wondering why I didn't include $\mathbb{A}$ four boxes ago when I wrote

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$$

Let's see if we can figure out where $\mathbb{A}$ would go, as a thought exercise. We know that all rational numbers are algebraic, so $\mathbb{A} \supseteq \mathbb{Q}$. We should ask if there are any algebraic numbers that are not rational, and of course we've all learned long ago (hopefully in middle school) that numbers like $\sqrt{2}$ and $\sqrt{5}$ are irrational. Since we proved they were algebraic, then we know $\mathbb{Q}$ is a proper subset of $\mathbb{A}$.

Next, we have to compare $\mathbb{A}$ and $\mathbb{R}$. That can't be too bad, right?

While we can write $\mathbb{A}$ to represent the set of algebraic numbers, it is interesting to note that some numbers are real but not algebraic, whereas other numbers are algebraic and not real. Moreover, some numbers are neither algebraic nor real, and some numbers (such as the integers and the rationals) are both algebraic and real.

Since this is very bad news for comparing $\mathbb{A}$ and $\mathbb{R}$, you should demand evidence from me of this fact. Normally, I like to convince you of everything I say, but that is not possible right now.

We will explore this matter further in the next two boxes.

First, let's list what we know.

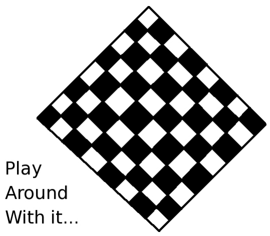
- Numbers like $\sqrt{-5} = i\sqrt{5}$ or $\sqrt{-2} = i\sqrt{2}$ are clearly not real. (The square root of a negative number is never real.
- Yet, $\sqrt{-5} = i\sqrt{5}$ and $\sqrt{-2} = i\sqrt{2}$ are algebraic, being the roots of $x^2 + 5 = 0$ and $x^2 + 2 = 0$.
- Rational numbers are both real and algebraic.
- These are three facts that we have shown rigorously.

Numbers like π and e are clearly real numbers. Yet, I stated (without any evidence) that these are not algebraic. The proof is very difficult, and I'm afraid that you'll have to trust me on that.

Numbers like $i\pi$ or ie are clearly not real, because they are $i = \sqrt{-1}$ times a real number. They are not algebraic either, as it turns out, but the proof is too difficult for this book.

If you are willing to accept these two additional facts from the previous box, then we can make four conclusions.

- $\sqrt{-5} = i\sqrt{5} \in \mathbb{A}$ and $\sqrt{-5} = i\sqrt{5} \notin \mathbb{R}$
- $\pi \notin \mathbb{A}$ and $\pi \in \mathbb{R}$
- $3/2 \in \mathbb{A}$ and $3/2 \in \mathbb{R}$
- $i\pi \notin \mathbb{A}$ and $i\pi \notin \mathbb{R}$



1-5-32

Just to be fair, I think it is right to tell you that the following questions are very good exam problems.

- Give a number that is algebraic, but not real.
- Give a number that is real, but not algebraic.
- Give a number that is both algebraic and real.
- Give a number that is neither algebraic nor real.

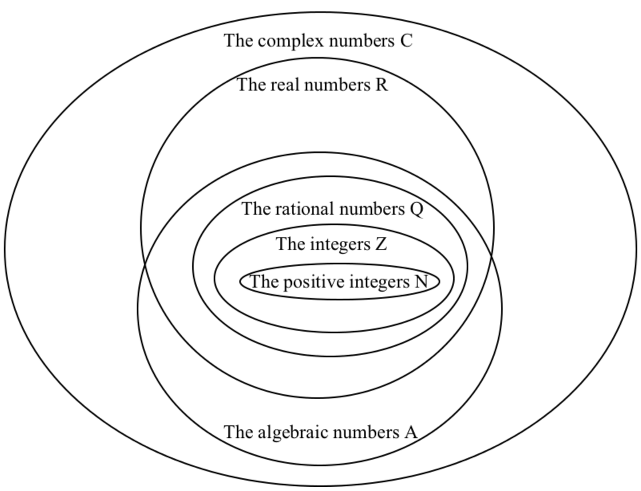
Of course, there are infinitely many answers to each of those questions, so I can't give you a list of answers!

I've shown evidence that it is not the case that $\mathcal{A} \subseteq \mathcal{R}$. I've stated (without proof) that $\mathcal{A} \supseteq \mathcal{R}$. Now we turn to $\mathbb{C}$.

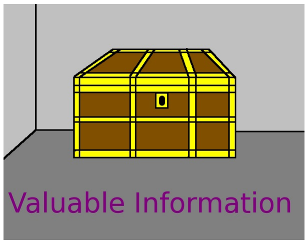
As it turns out, "the fundamental theorem of algebra" guarantees that all polynomials with integer coefficients have roots in $\mathbb{C}$. That remains true regardless if the coefficients come from $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{A}$, $\mathbb{R}$, or $\mathbb{C}$, by the way.

Since all polynomials (with integer coefficients) have their roots in $\mathcal{C}$, and since a number is algebraic if and only if it is the root of a polynomial (with integer coefficients), then $\mathbb{A} \subseteq \mathcal{C}$. Because of numbers like $i\pi$ or ie, it is a proper subset, so we can write $\mathbb{A} \subsetneq \mathcal{C}$.

This means that the final picture looks like the one in the next box.



The diagram on the left shows the relationship among $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{A}$, $\mathbb{R}$, and $\mathbb{C}$.

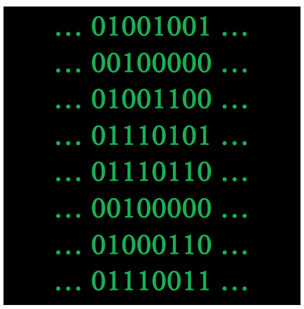


You'll recall that if a number is not rational, we call it irrational. You might wonder if there is a similar name for numbers that are not algebraic.

Real and complex numbers that are not algebraic are called *transcendental numbers*. I don't like this word at all. It sounds like one is speaking of mysticism, medieval alchemy, or science fiction. Many in the computer science community agree, and prefer to say "π is non-algebraic" rather than "π is transcendental." Mathematicians tend to use "transcendental."

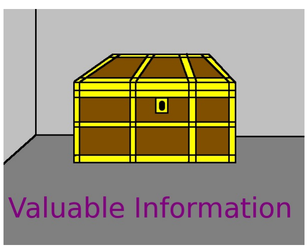
If you're curious why the word "transcendental" was chosen, it has to do with the degree of algebraic numbers. One way to think of non-algebraic numbers is that they have an algebraic degree of ∞. In that sense, they "transcend" the concept of degree.

However, there is no mathematical meaning to the phrase "an algebraic degree of ∞" so one should avoid it.



While our discussion of algebraic numbers has hopefully been clear and interesting, you might be wondering why this is connected to computer algebra systems. Many algebraic numbers can be represented exactly in computer algebra software, just like your precalculus teacher or algebra teacher would handle $\sqrt{3}/2$ instead of writing that as a decimal.

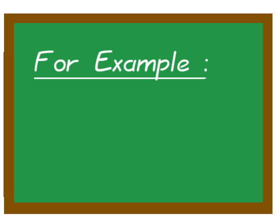
Because no computer has infinite memory, whenever a computer stores an irrational number, it cannot store all the digits, because there are infinitely many. Thus, some rounding error is incurred. While that is acceptable in some cases, it is not acceptable in others. Therefore, computer algebra systems like Sage, Magma, or Maple (but not MATLAB), will try to represent irrational numbers exactly. This is in stark contrast to ordinary programming languages, such as Python, C, C++, Java, FORTRAN, or Perl which use approximations, called "`float`" or "`double`."



In fact, all algebraic numbers of degree 4 or lower can always be written as formulas. (We'll see a degree 4 case over the next two boxes.) Also, infinitely many algebraic numbers of every higher degree can be written as formulas.

Starting at degree 5, there are some algebraic numbers that cannot be written as a formula, a consequence of the Abel–Ruffini Theorem. That's named for Niels Henrik Abel (1802–1829) and Paolo Ruffini (1765–1822). Computer algebra systems can still work with those algebraic numbers that cannot be written as formulas, but it is very unpleasant and inconvenient. So for all practical purposes, there is a wall between the algebraic numbers that are fourth-degree or lower, and those that are fifth-degree or higher.

By the way, Ruffini also gave us the variant of polynomial long division called "synthetic division."



1-5-33

I can make the previous box a lot more concrete with an example. Consider the polynomial

$$p(x) = x^4 + x^3 + x^2 + x - 1$$

Surely its roots are algebraic numbers, because this polynomial has integer coefficients. It turns out that two roots are real, and two roots are complex. As mere decimal approximations, those roots are

$-1.29064880\cdots$

$0.518790063\cdots$

$-0.114070631\cdots - (i)(1.21674600\cdots)$

$-0.114070631\cdots + (i)(1.21674600\cdots)$

Now let's see how Sage will view these as algebraic numbers. The code is given on the next page.

```
p(x) = x^4 + x^3 + x^2 + x - 1

answers = solve( p(x) == 0, x )

# Like many programming languages, Python and Sage number their arrays
# from 0 and not from 1. So the four answers are called answers[0], [1], [2], and [3]
# instead of answers[1], [2], [3], and [4]
#
show(answers[0])
show(answers[1])
show(answers[2])
show(answers[3])
```

Using the code above, Sage will solve the polynomial from the previous box, exactly and not numerically.

Sage's response is as follows:

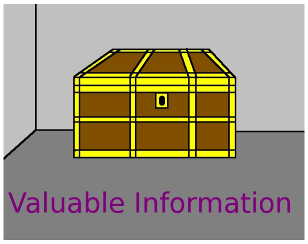
$$x = -\frac{1}{12}\sqrt{\frac{36\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{2}{3}}-15\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}-56}{\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}}} - \frac{1}{2}\sqrt{-\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}+\frac{14}{9\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}}+\frac{15}{2\sqrt{\frac{36\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{2}{3}}-15\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}-56}{\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}}}}-\frac{5}{6}} - \frac{1}{4}$$

$$x = -\frac{1}{12}\sqrt{\frac{36\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{2}{3}}-15\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}-56}{\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}}} + \frac{1}{2}\sqrt{-\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}+\frac{14}{9\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}}+\frac{15}{2\sqrt{\frac{36\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{2}{3}}-15\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}-56}{\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}}}}-\frac{5}{6}} - \frac{1}{4}$$

$$x = \frac{1}{12}\sqrt{\frac{36\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{2}{3}}-15\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}-56}{\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}}} - \frac{1}{2}\sqrt{-\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}+\frac{14}{9\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}}-\frac{15}{2\sqrt{\frac{36\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{2}{3}}-15\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}-56}{\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}}}}-\frac{5}{6}} - \frac{1}{4}$$

$$x = \frac{1}{12}\sqrt{\frac{36\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{2}{3}}-15\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}-56}{\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}}} + \frac{1}{2}\sqrt{-\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}+\frac{14}{9\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}}-\frac{15}{2\sqrt{\frac{36\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{2}{3}}-15\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}-56}{\left(\frac{1}{18}\sqrt{563}\sqrt{3}+\frac{65}{54}\right)^{\frac{1}{3}}}}}-\frac{5}{6}} - \frac{1}{4}$$

As you can see, Sage has successfully written all four roots as formulas.



What the Abel-Fubini theorem says is that there is no quintic formula—there is no formula to give you the roots of a general fifth degree polynomial.

To be very precise, there is no function built up of addition, subtraction, multiplication, division, roots, exponents, logarithms, trigonometric functions and inverse trigonometric functions which is capable of giving you a root for any quintic polynomial that you choose to challenge it with. We can even include the hyperbolic cousins of the trigonometric functions and inverse trigonometric functions, which you might or might not have been taught about (hypsin, hypcos, hyptan, ...).

It is possible to write a formula for producing a root for an arbitrary quintic polynomial using tools called "elliptic functions" but these are very hard to work with.

We'll discuss Abel's biography in the next box.

Niels Henrik Abel (1802–1829) is mostly known for the Abel-Fubini theorem. His life's story is most interesting. When Norway broke off from Denmark in 1814 and joined Sweden, Niels Abel's father, Soren Abel, was elected a member of the new Norwegian Parliament. The parliament met in the great hall of the cathedral school in the city called Christiana (the city was renamed Oslo in 1925). Soren Abel was so impressed with the school that he decided his eldest son, Hans Mathias Abel, should study there.

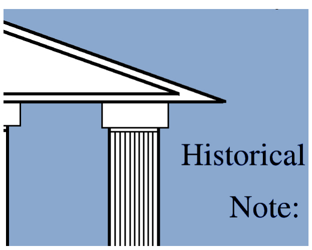
However, Hans was very depressed at the thought of leaving home, and the family decided to send Niels Abel instead, who was 13. The mathematics tutor there, Bernt Michael Holmboe (1795–1850), realized Abel's talent and mentored him.

Continuing with the previous box, meanwhile Soren Abel got entangled into two public disputes—one theological, and one political. By 1818, his political career was ruined. Niels was only 16 at the time, and his mentor Holmboe arranged funds for him to continue attending school. Meanwhile, the father, Soren Abel, began drinking heavily, and died in 1820, leaving the family penniless.

Holmboe arranged a scholarship for Niels Abel, and by the time he arrived at the Royal Frederick University (now called The University of Oslo) in 1821 at the age of 19, Niels Abel started reading mathematics in the university library. That same year, he thought he found the quintic equation. He asked several well-known mathematicians to check it, and they found no error. Neils Abel himself found the mistake when pressed to carry out a numerical example by Carl Ferdinand Degen (1766–1825), one of the mathematicians asked to check the proof.

How ironic that Niels Abel would later prove that no such quintic equation could exist! Amazingly, he graduated in 1822, rather soon after arrival on campus in 1821.

We will continue the story in the next box.



Niels Abel attempted to begin a scholarly career in the usual way—traveling to European universities to study, to meet famous mathematicians, to publish his work, and perhaps get a PhD. However, he was severely limited by his poverty. (Recall, the death of his father had left him penniless.) Some professors supported him financially, and one professor let him live in the attic of his home.

Much of Neils Abel's time was spent applying for scholarships, funding to travel, and looking after his siblings. Niels Abel did finally make it to Europe and travelled around, but he contracted tuberculosis in Paris. He returned to Norway, and died in 1829 at the age of 26. Two days after he died, a letter arrived appointing him to a professorship at the University of Berlin.

Most of Abel's discoveries would have been lost, but his old tutor, Bernt Michael Holmboe (1795–1850) published all of Abel's writings on mathematics ten years later, except for one result that I'll describe in the next box.

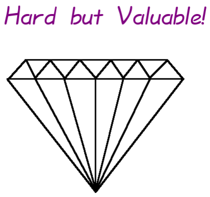
It is not surprising that dire poverty hampered his career. What is surprising is that the Norwegian community failed to recognize his brilliance, while being fully aware that he completed his undergraduate degree in mathematics in one year. However, in the 20th century, the Kingdom of Norway would make amends for this tragedy, as we will see on Page 225 of the module "Fermat's Last Theorem and Some Famous Unsolved Problems."

Abel also wrote a paper about functions which cannot be symbolically integrated. In other words, functions $f(x)$ such that there is no way to write down a function built up of addition, subtraction, multiplication, division, roots, exponents, logarithms, trigonometric functions, inverse trigonometric functions, hyperbolic trigonometric functions, and hyperbolic inverse trigonometric functions such that the derivative of the function you write is $f(x)$. This paper was unfortunately lost while being checked, and was never rewritten.

The most famous example of such an $f(x)$ would be Fresnel's integral

$$f(x) = \cos x^2$$

which is involved in how light and shadows behave when they strike solid objects. These integrals also have applications in the design of highways and roller-coasters.



In the tail end of this module, we're going to look at two proofs. In calculus classes, the proofs are somewhat important but they aren't the core activity of the course. In Discrete Mathematics, this changes. Of course, Discrete Mathematics is not entirely about proofs. There are many interesting problems that we can solve with Discrete Mathematics. However, the proofs are much more important in Discrete Mathematics than they were in calculus class. The proofs simply cannot be skipped.

The first proof is a super-classic beautiful proof, dating back to the Ancient Greeks. We will prove that $\sqrt{2}$ is irrational. The second proof is also a classic and beautiful proof, but classic in a different way. It is a favorite among graduate students in math, talking to each other in the hallways.

The second proof will show that there exists some irrational number, raised to an irrational power, that equals a rational number. Yet, the proof gives no example of such a number, despite the fact that this existence has been proven beyond any shadow of a doubt.

It is very clear that

$$\sqrt{2} = 1.41421356\cdots$$

is not an integer. Another way to see this is to write $1 < 2 < 4$, which means that $\sqrt{1} < \sqrt{2} < \sqrt{4}$ or more simply $1 < \sqrt{2} < 2$. Since there is no integer strictly between 1 and 2, for sure, $\sqrt{2}$ is not an integer.

Now let's prove that $\sqrt{2}$ is not rational. We will do that in the next box, by using "proof by contradiction."

Claim: $\sqrt{2}$ is irrational.

Proof: Assume that $\sqrt{2}$ is rational. This means that we can write $\sqrt{2}$ as a ratio a/b, where a and $b \neq 0$ are integers, and a/b is reduced to lowest terms.

$$\begin{aligned} \sqrt{2} &= a/b \\ 2 &= a^2/b^2 \\ 2b^2 &= a^2 \end{aligned}$$

The value of $2b^2$ is even, because it is an integer (b^2) times two. Since $a^2 = 2b^2$, this means that a^2 is even also.

Recall that an odd number times an odd number is odd. So if a were odd, then a^2 would be odd. Since a^2 is even, this means that a is even.

Since a and b are both even, then the fraction a/b is not reduced to lowest terms (because the numerator and denominator could both be divided by two). This is a contradiction!

Therefore, our initial assumption must have been false—it is not the case that $\sqrt{2}$ is rational. In conclusion, $\sqrt{2}$ is irrational. ■

We're going to have a little bit of an excursion now. In the next twelve boxes, we're going to explore some more advanced ideas about rational versus irrational numbers.

While theoretical, this is important, because it will give you an idea of why there is no easy test to see if a number is rational or irrational.

From time to time, students ask me for an easy test to see if a number is rational or irrational. No such test exists. To convince you of that, permit me to tell you that there are several numbers where it is currently *unknown* whether or not the number is rational or irrational. Here is a partial list:

$$\left\{ e\pi,\ \frac{\pi}{e},\ 2^e,\ e^e,\ e^{e^e},\ \pi^e,\ \pi^{\sqrt{2}},\ \ln\pi \right\}$$

Keep in mind that this could change at any moment. Someone could publish a proof of the rationality or irrationality of any of the above numbers, between when I write this and when you read it. The above list was correct as of August of 2017, according to both MathWorld and Wikipedia.

It is easy to prove that there exists some irrational number, raised to an irrational power, that equals a rational number. Yet, that classic proof does not provide any examples of such numbers.

As it turns out, the proof revolves around $\sqrt{2}^{\sqrt{2}}$, so let's take a moment to talk about that number in the next box.

Sometimes a gifted student will question me as to what $\sqrt{2}^{\sqrt{2}}$ really means. We know what integer exponents mean, like

$$4^3 = (4)(4)(4) = 64 \qquad \text{and} \qquad 8^4 = (8)(8)(8)(8) = 4096$$

and we know what rational exponents mean, like

$$\left(\frac{125}{8}\right)^{4/3} = \left(\left(\frac{125}{8}\right)^{1/3}\right)^4 = \left(\sqrt[3]{\frac{125}{8}}\right)^4 = \left(\frac{\sqrt[3]{125}}{\sqrt[3]{8}}\right)^4 = \left(\frac{5}{2}\right)^4 = \frac{5^4}{2^4} = \frac{625}{16}$$

or perhaps alternatively

$$\left(\frac{49}{9}\right)^{5/2} = \left(\left(\frac{49}{9}\right)^{1/2}\right)^5 = \left(\sqrt{\frac{49}{9}}\right)^5 = \left(\frac{\sqrt{49}}{\sqrt{9}}\right)^5 = \left(\frac{7}{3}\right)^5 = \frac{7^5}{3^5} = \frac{16,807}{243}$$

but what does $\sqrt{2}^{\sqrt{2}}$ mean?

Because of the fact that

$$\sqrt{2} \approx 1.4142135623730950488016887242096980785696718753 76\cdots$$

you can think of $\sqrt{2}^{\sqrt{2}}$ as a limit

$$1^1,\ 1.4^{1.4},\ 1.41^{1.41},\ 1.414^{1.414},\ 1.4142^{1.4142},\ 1.41421^{1.41421},\ \ldots$$

The sequence and limit of the previous box will placate some students, but not others. After all, the limits that we take in our calculus classes usually involve a rational function (a ratio of two polynomials) or a polynomial inside of a square root. Maybe some might involve a trigonometric function or an exponential. We expect to see a formula when we take a limit.

In the next box, we will write this limit as the limit of a formula, rather than as a sequence of decimal numbers.

Looking at the sequence of the previous box, we can think of it as a formula:

$$a_j = \text{floor}\left((\sqrt{2})(10^j)\right)(10^{-j})$$

which is just a formula after all, like the formulas whose limits you have taken in calculus class. By the way, floor(x) is the formal mathematical way of saying "round down."

Many students will find the formula for a_j very confusing. I will demonstrate its meaning using Sage in the next box. However, this is a side topic, and please do not panic if this fails to make any sense. If you understand

$$1^1,\ 1.4^{1.4},\ 1.41^{1.41},\ 1.414^{1.414},\ 1.4142^{1.4142},\ 1.41421^{1.41421},\ \ldots$$

as a pattern, then we are fine.

Let's explore that pattern with Sage. The following Sage code will compute for us the limit discussed in this box, which will evaluate to $\sqrt{2}^{\sqrt{2}}$.

```
for j in range(0, 13):
    # this means j will go from 0 to 12, inclusive, but not 13
    # j is the number of decimal places from sqrt(2) used
    x = floor(sqrt(2)*10^j)*10^(-j)
    print N(x), "^", N(x),
    print "=", N(x^x)
    # N(stuff) means "a decimal approximation of stuff"

print "Final Answer:"
print N(sqrt(2)), "^", N(sqrt(2)), "="
print N(sqrt(2)^sqrt(2), digits=50)
```

That code produces the following output:

```
1.00000000000000 ^ 1.00000000000000 = 1.00000000000000
1.40000000000000 ^ 1.40000000000000 = 1.60169289820221
1.41000000000000 ^ 1.41000000000000 = 1.62330059876299
1.41400000000000 ^ 1.41400000000000 = 1.63205753532488
1.41420000000000 ^ 1.41420000000000 = 1.63249710541261
1.41421000000000 ^ 1.41421000000000 = 1.63251908823666
1.41421300000000 ^ 1.41421300000000 = 1.63252568316411
1.41421350000000 ^ 1.41421350000000 = 1.63252678232229
1.41421356000000 ^ 1.41421356000000 = 1.63252691422134
1.41421356200000 ^ 1.41421356200000 = 1.63252691861797
1.41421356230000 ^ 1.41421356230000 = 1.63252691927747
1.41421356237000 ^ 1.41421356237000 = 1.63252691943135
1.41421356237300 ^ 1.41421356237300 = 1.63252691943794
Final Answer:
1.41421356237310 ^ 1.41421356237310 =
1.6325269194381528447734953810247196020791088570531
```

In conclusion,

$$\sqrt{2}^{\sqrt{2}} \approx 1.632526919438152844773495381024719602079108857053\cdots$$

is just a number on the real line. We can define it as a limit, we can work with a decimal approximation of it, or we can work with it symbolically, as we will do in the next box.

Now I'd like to show you the proof that there exists an irrational number raised to an irrational power that equals a rational number. This proof was described to me by Dr. George Brown, who lectured at UW Stout during the 2012–2013 academic year. While he definitely did not discover this proof, he said it was his favorite proof, and to honor him, we will call $\sqrt{2}^{\sqrt{2}}$ by the name "b" for Dr. Brown.

By the way, this proof is not terribly hard but it certainly isn't easy. Please do not be upset if you cannot understand it. For now, give it an honest try. If you cannot understand it, then return to it at the end of the course, and it will probably make sense at that time. On the other hand, if you do understand it, then please accept my congratulations.

Claim: there exists an irrational number raised to an irrational power that equals a rational number.

Proof: Consider two cases: either $b = \sqrt{2}^{\sqrt{2}}$ is rational or irrational.

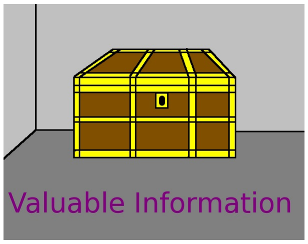
Case 1: Suppose b is rational. Since $\sqrt{2}$ is irrational, then $\sqrt{2}^{\sqrt{2}} = b$ is an example of an irrational number, raised to an irrational power, that equals a rational number.

Case 2: Suppose b is irrational. Consider $b^{\sqrt{2}}$. Observe,

$$b^{\sqrt{2}} = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2})(\sqrt{2})} = \sqrt{2}^{\sqrt{(2)(2)}} = \sqrt{2}^{\sqrt{4}} = \sqrt{2}^{2} = 2$$

Since b is irrational and $\sqrt{2}$ is irrational, but 2 is rational, there exists an example of an irrational number, raised to an irrational power, that equals a rational number.

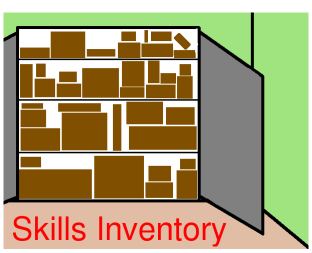
In both cases, there exists an example of an irrational number, raised to an irrational power, that equals a rational number. ∎



You might be thinking "What are the 'take-away concepts' of the last eleven boxes?" What is it, exactly, that I wanted you to learn from this excursion?

1. There is no easy test for whether or not a number is rational or irrational.
2. There are some numbers that have not yet been proven to be rational or irrational.
3. There are some straightforward questions in math whose answer is not yet known.
4. I hope you have some further insight into what $\sqrt{2}^{\sqrt{2}}$ means, but maybe you understood that already.
5. I'd be delighted if you understood the proof that there exists an irrational number raised to an irrational power that equals a rational number.
6. However, if you didn't understand it, then that's no problem at all. You'll be able to read such proofs when the course is over.

By the way, an explicit example of an irrational number, to an irrational power, equaling a rational number, will be given on Page 235 of the module "Fermat's Last Theorem and Some Famous Unsolved Problems."



This module is now complete. Here is a list of some of the things that you learned during this module.

- The symbols for some very famous sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ as well as $\mathbb{A}$.
- Why researchers in discrete mathematics no longer say "the natural numbers."
- Set-builder notation.
- Computing the solution set of an equation.
- Computing the truth set of an inequality.
- The power-set operation.
- The set-subtraction operation.
- The Cartesian Product operation, including exponents of sets.
- What the algebraic numbers are.
- What the degree of an algebraic number means.
- Finding polynomials that determine the degree of an algebraic number.
- We had the vocabulary terms: algebraic numbers, Cartesian Product, complex numbers, degree (of an algebraic number), integers, irrational, minimal polynomial (of an algebraic number) rational, root of a polynomial, quotient, solution set, truth set, and transcendental numbers.

On Page 151, you were asked to determine if the following numbers are rational or irrational.

- 17 can be written as 17/1, so it is rational.
- 2/3 is a ratio of two integers, so it is rational by definition.
- $\sqrt{2} = 1.41421356\cdots$. Irrational, because whenever the square root of an integer is not an integer, it is irrational.
- 5/4 is a ratio of two integers, so it is rational by definition.
- $\sqrt{3}/2$ is interesting. We'll describe this in the next box.
- 1/2 is a ratio of two integers, so it is rational by definition.
- 0 can be written as 0/1, so it is rational.

Claim: $\sqrt{3}/2$ is irrational.
Proof: Assume $\sqrt{3}/2$ is rational. That means we can write it as $\sqrt{3}/2 = a/b$, for some integers a and $b \neq 0$. However, observe

$$\sqrt{3} = 2\frac{\sqrt{3}}{2} = 2\frac{a}{b} = \frac{2a}{b}$$

which would imply that we have written $\sqrt{3}$ as the ratio of two integers, $2a$ and b. This would imply $\sqrt{3} = 1.73205080\cdots$ is rational, but we know that whenever the square root of an integer is not an integer, it is irrational. This is a contradiction (because $\sqrt{3}$ cannot be both rational and irrational).

Therefore, our initial assumption, that $\sqrt{3}/2$ is rational, must be false. In conclusion, $\sqrt{3}/2$ is irrational. ■

On Page 152, you were asked to determine if the following numbers are rational or irrational.

- $\sin 90° = 1$, which is rational.
- $\sin 60° = \sqrt{3}/2$, which we saw, in the previous box, was irrational.
- $\sin 45° = \sqrt{2}/2$. With the same argument as $\sqrt{3}/2$ from the previous box, it is irrational.
- $\sin 30° = 1/2$, a ratio of two integers, so it is rational by definition.
- $4^{5/2} = (4^5)^{1/2} = \sqrt{1024} = 32 = 32/1$ is rational.
- (Equivalently, $4^{5/2} = (4^{1/2})^5 = (\sqrt{4})^5 = 2^5 = 32 = 32/1$ is rational.)
- $2^{5/2} = (2^5)^{1/2} = \sqrt{2^5} = \sqrt{32} = 5.65685424\cdots$ is clearly the square root of an integer, but is not an integer. Therefore, it is irrational.
- $\sqrt{20} - 2\sqrt{5} = \sqrt{(4)(5)} - 2\sqrt{5} = \sqrt{4}\sqrt{5} - 2\sqrt{5} = 2\sqrt{5} - 2\sqrt{5} = 0$ is rational.

For the last one, most of my students would just plug $\sqrt{20} - 2\sqrt{5}$ into their calculator, see 0, and say, "rational."

Here are the rosters for the solution sets I asked you to compute on Page 157.

- $\{x \in \mathbb{R} \,|\, x^3 = x\} = \{-1, 0, 1\}$
- $\{x \in \mathbb{R} \,|\, x^2 + x + 1 = 0\} = \{\}$
- $\{x \in \mathbb{C} \,|\, x^2 + x + 1 = 0\} = \left\{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\right\}$
- $\{x \in \mathbb{Z} \,|\, (x-2)(2x-3) = 0\} = \{2\}$
- $\{x \in \mathbb{Q} \,|\, (x-2)(2x-3) = 0\} = \{3/2, 2\}$

Here are the rosters for the truth sets I asked you to compute on Page 158.

- $\{x \in \mathbb{Z} \,|\, -3 < x < 7\} = \{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$
- $\{x \in \mathbb{Z}^+ \,|\, -3 < x < 7\} = \{1, 2, 3, 4, 5, 6\}$
- Remember, $\mathbb{Z}^+$ means "the positive integers," unambiguously. We explicitly exclude zero when we say "positive," because 0 is neither positive nor negative. If I had written $\{x \in \mathbb{N} \,|\, -3 < x < 7\}$, then there would be an ambiguity. Some textbooks include $0 \in \mathbb{N}$, but some textbooks do not. For this reason, researchers and advanced textbooks simply avoid the symbol $\mathbb{N}$ and avoid the phrase "the natural numbers."
- $\{x \in \mathbb{Z} \,|\, -\pi \leq x \leq \pi\} = \{-3, -2, -1, 0, 1, 2, 3\}$
- $\{x \in \mathbb{Z}^+ \,|\, -\pi \leq x \leq \pi\} = \{1, 2, 3\}$
- $\{x \in \mathbb{Z}^+ \,|\, 5x - 19 < 2\} = \{1, 2, 3, 4\}$

Here is the answer to the slightly challenging problem from Page 158.

$$\{x \in \mathbb{Z}^+ \,|\, x > 2\} \cap \left\{x \in \mathbb{R} \,\middle|\, x < \sqrt{61}\right\} = \{3, 4, 5, 6, 7\}$$

Here are the solutions to the question about power sets, as given on Page 159.

1. $P(\{5,6,7\}) = \{\{\};\ \{5\};\ \{6\};\ \{7\};\ \{5,6\};\ \{5,7\};\ \{6,7\};\ \{5,6,7\}\}$
2. $P(\{8,9\}) = \{\{\};\ \{8\};\ \{9\};\ \{8,9\}\}$
3. $P(\{0\}) = \{\{\};\ \{0\}\}$
4. $P(\{\}) = \{\{\}\}$

On Page 160, you were asked to replace the "?" in each of four statements with one of four possible symbols: $\{\subsetneq, \supsetneq, =, \in\}$

- $\{W, X\} \supsetneq \{W\}$
- $\{W, X\} \subsetneq \{W, X, Y\}$
- $\{\{W, X\};\ \ \{W\}\ \ \} \subsetneq P(\{W, X, Y\})$
- $\{W, X\} \in P(\{W, X, Y\})$

Here are the solutions to the question about set subtraction, as given on Page 160.

1. $\mathcal{X} - \mathcal{Z} = \{1, 3\}$
2. $\mathcal{Y} - \mathcal{X} = \{2, 4\}$
3. $\mathcal{Y} - \mathcal{Z} = \{1, 2, 3\}$
4. $\mathcal{Z} - \mathcal{X} = \{4, 6\}$
5. $\mathcal{Z} - \mathcal{Y} = \{5, 6, 7\}$
6. $\mathcal{Z} - \mathcal{Z} = \{\}$

Let's discussion the question (from Page 161) of whether or not

$$\mathcal{A} - \mathcal{B} = \mathcal{B} - \mathcal{A} \text{ is true or false}$$

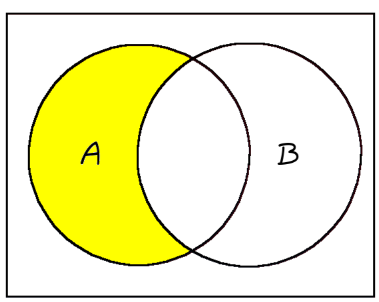
Since in our example, we saw that $\mathcal{X} - \mathcal{Z} \neq \mathcal{Z} - \mathcal{X}$ then surely it is obvious that $\mathcal{A} - \mathcal{B} \neq \mathcal{B} - \mathcal{A}$ is not true in general for all sets.

However, if $\mathcal{A}$ and $\mathcal{B}$ are disjoint, then

$$(\mathcal{A} - \mathcal{B} = \{\}) \text{ and } (\mathcal{B} - \mathcal{A} = \{\}) \text{ therefore } (\mathcal{A} - \mathcal{B} = \mathcal{B} - \mathcal{A})$$

Here is the solution to the question that asks you to shade a 2-circle Venn Diagram (from Page 162), in order to show what $\mathcal{A} - \mathcal{B}$ means.

My diagram looks like this:



On Page 162, you were asked to insert one of the four symbols: $\oplus, \cap, \cup, -$ in place of the question mark for nine formulas from set theory. Here are the solutions.

- $\{A, B, C, D\} \oplus \{A, C, E\} = \{B, D, E\}$
- $\{A, C, E\} - \{B, C\} = \{A, E\}$
- $\{A, B, C, D, E\} \cap \{\} = \{\}$
- $\{A, C, E\} \cap \{C\} = \{C\}$
- $\{A, B, C, D\} - \{A, C, E\} = \{B, D\}$
- $\{A, C, E\} \cup \{C\} = \{A, C, E\}$
- $\{A, B, C, D\} \cup \{A, C, E\} = \{A, B, C, D, E\}$
- $\{A, C, E\} \oplus \{B, C\} = \{A, B, E\}$
- $\{A, B, C, D\} \cap \{A, C, E\} = \{A, C\}$

The solutions to the Cartesian Product question (from Page 164) are given below.

1. $\{0, 3\} \times \{1, 2, 4, 8\} = \{(0, 1);\ (0, 2);\ (0, 4);\ (0, 8);\ (3, 1);\ (3, 2);\ (3, 4);\ (3, 8)\}$
2. $\{1, 2, 3, 4, 5\} \times \{0\} = \{(1, 0);\ (2, 0);\ (3, 0);\ (4, 0);\ (5, 0)\}$
3. $\{a, b, c\} \times \{x, y, z\} = \{(a, x);\ (a, y);\ (a, z);\ (b, x);\ (b, y);\ (b, z);\ (c, x);\ (c, y);\ (c, z)\}$
4. $\{a, b\} \times \{b, c\} = \{(a, b);\ (a, c);\ (b, b);\ (b, c)\}$

On Page 164, you were asked to find a formula for size during set subtraction. The formula clearly can't be

$$\#(\mathcal{A} - \mathcal{B}) = (\#\mathcal{A}) - (\#\mathcal{B})$$

because of simple examples like

$$\{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\} \qquad \text{but} \qquad 3 - 3 \neq 2$$

As it turns out, the answer is

$$\#(\mathcal{A} - \mathcal{B}) = \#\mathcal{A} - \#(\mathcal{A} \cap \mathcal{B})$$

However, this can be written as

$$\#(\mathcal{A} - \mathcal{B}) = \#(\mathcal{A} \cup \mathcal{B}) - \#\mathcal{B}$$

Here are the solutions to the questions about raising sets to powers (from Page 168).

- If $\mathcal{C} = \{1, 0, -1\}$, then

$$\mathcal{C}^2 = \{(1,1);\ (1,0);\ (1,-1);\ (0,1);\ (0,0);\ (0,-1);\ (-1,1);\ (-1,0);\ (-1,-1)\}$$

- If $\mathcal{D} = \{0, 1\}$, then

$$\mathcal{D}^3 = \{(0,0,0);\ (0,0,1);\ (0,1,0);\ (0,1,1);\ (1,0,0);\ (1,0,1);\ (1,1,0);\ (1,1,1)\}$$

Here are the answers for the question about algebraic numbers from Page 171.

- $\sqrt[19]{101}$ is a root of $x^{19} - 101 = 0$.
- $\frac{1+\sqrt{5}}{2}$ is a root of $x^2 - x - 1 = 0$.
- $1/\sqrt{5} = \sqrt{5}/5$ is a root of $25x^2 - 5 = 0$.

$$\text{Note: } \frac{1}{\sqrt{junk}} = \frac{\sqrt{junk}}{junk}$$

- $\sqrt{3}/2$ is a root of $4x^2 - 3 = 0$.
- Of course, $\sin 60° = \sqrt{3}/2$, so clearly $\sin 60°$ is a root of $4x^2 - 3 = 0$.

Question: For what positive integers z is the number $\sqrt[z]{2}$ an algebraic number?

Answer: All of them.

That's because the number $\sqrt[z]{2}$ is algebraic for any positive integer z, because it is a root of the polynomial $x^z - 2 = 0$.

On Page 173, you were asked for polynomials to justify the following statements.

- The algebraic number $\sqrt[6]{4}$ is of the third degree. [Answer: $x^3 - 2 = 0$.]
- The algebraic number $\sqrt[8]{4}$ is of the fourth degree. [Answer: $x^4 - 2 = 0$.]
- The algebraic number $\sqrt[6]{27}$ is of the second degree. [Answer: $x^2 - 3 = 0$.]
- The algebraic number 5 is of the first degree. [Answer: $x - 5 = 0$.]

You might end up with different polynomials than I do. Just make sure that your answer polynomial is of the correct degree, and that the number in question (either $\sqrt[6]{4}$ or $\sqrt[8]{4}$) is actually a root of the polynomial which you wrote down.

Permit me to show you what I mean in the next box.

Here's how you can make sure that your polynomial has the algebraic number in question as a root. Looking at the previous box, here is how we would check the first part.

$$x^3 - 2 = \left(\sqrt[6]{4}\right)^3 - 2 = (4)^{3/6} - 2 = (4)^{1/2} - 2 = 2 - 2 = 0$$

Here's how we would check the third part.

$$x^2 - 3 = \left(\sqrt[6]{27}\right)^2 - 3 = (27)^{2/6} - 3 = (27)^{1/3} - 3 = 3 - 3 = 0$$

Some students prefer to use decimals, and that's fine. However, it is slower, because you have to operate your calculator if you use decimals.