Module 5.1: A Workbook on Independence and Repetition

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1 Theory

- Technically, the definition of independence is

  The events A and B are independent if and only if $Pr\{A \cap B\} = Pr\{A\}Pr\{B\}$

- However, we use this definition in two distinct ways.
  - Sometimes, you want to test if A and B are independent. Then check to see if $Pr\{A \cap B\} = Pr\{A\}Pr\{B\}$ is true.
    - If they are exactly equal, then you know A and B are independent for sure.
    - If they are far apart, then you know A and B are definitely not independent.
    - If they are close, then there are tests from advanced courses in statistics that will cover how to handle this situation, but we cannot discuss it at this time.
  - Sometimes, you know A and B are independent. In that situation, $Pr\{A \cap B\} = Pr\{A\}Pr\{B\}$ is a handy formula.
  - Another handy fact is that if A and B are independent, then for sure, A^c and B^c are independent.
- Of course, if event A occurs with probability p, and you make two independent attempts at event A, you get event A both times with probability $(p)(p) = p^2$.
- That last bullet generalizes to more than 2 repetitions. This gives us the two repetition formulas:
  - If you have n independent repetitions of an attempt of event A, and if each attempt succeeds with probability p, then the probability of all n attempts succeeding is p^n.
  - If you have n independent repetitions of an attempt of event A, and if each attempt succeeds with probability p, then the probability of all n attempts failing is $(1-p)^n$.

As you can see, if I have n independent repetitions of an event, the last two bullets give me the probability of n success and 0 failures, or 0 successes and n failures. What about situations in between these two extremes? That's the topic of the next module—The Binomial Distribution Formula.

2 Questions

1. Suppose that on a given college campus, 95% of the students have consumed alcohol, 60% of the students have smoked marijuana, and 57% of the students have used both. Is the probability of a randomly selected student (from that campus) having smoked marijuana independent of a randomly selected student (from that campus) having consumed alcohol?

2. Let's imagine that among certain cheaply-made smartphones, the screens have flaws with probability 3%, and the batteries have flaws with probability 5%. Bob is going to assume that these flaws are independent, because there are two different companies supplying the parts. If a smartphone has no flaws, it is shipped out for sale. If it has one flaw, it is kept for refurbishing. If it has both flaws, it is tossed out. Compute the probability distribution for smartphones being shipped, refurbished, and tossed.

3. Continuing with the previous problem, suppose that 1% of the phones are found to have both flaws. Was Bob's assumption of independence correct? What is the new probability distribution?

Note: Do you remember, back in the module "A Formal Introduction to Probability Theory," when I told you that there were six common errors? I mentioned that one of them is hard to explain, deals with independence, and would be explained in this module. The previous two problems illustrate this common pitfall. It is a common mistake to assume independence when the events might not necessarily be independent.

Background: Some of us have heard of "cancer alley," the portion of the Mississippi river between Baton Rogue and New Orleans. This area had (at least for a time) an unusually high rate of cancer—and some estimated it to be nine times the national average. The movie *Erin Brockovich* also told the story of another area with a high density of cancer, but in California. The Centers for Disease Control use all sorts of mathematical techniques to detect these cancer "hotspots." In turn, identifying these locations can help uncover the underlying causes (such as industrial pollution). Then the problems can be remedied, preventing much unnecessary loss of life. It is interesting for us to explore the mathematics that underlies these investigative techniques. We will do that in the following problem.

(For the more human aspects of investigating cancer hotspots, it is worth noting that the movie *Erin Brockovich* is only 2 hours long and has won many awards.)

4. Imagine a small town, with a population of 9000 people. There is a petrochemical factory there, and 900 of the town's residents work there. There are 234 people in the town who have cancer, or who have had cancer, and 72 of them work at the factory. The CDC has dispatched an epidemiologist to this town to investigate. The factory's public relations team notes that *more than two thirds* of the cancer patients in the town are *not employed at the factory in any capacity whatsoever*. Moreover, 92% of the employees of the factory are cancer free. This might make a naïve person think that there is no link between the factory and cancer. The epidemiologist, of course, can use the mathematics of probability to see through this metaphorical cloud of smoke.

   Note: As is standard, we will use the phrase "has cancer" as an abbreviation for "has cancer, or has had cancer." We will use the phrase "is cancer free" as an abbreviation for "does not have cancer, nor has ever had cancer."

   (a) What is the probability that a random person in the town has cancer?

   (b) What is the probability that a random person in the town works for the factory?

   (c) What is the probability that a random person in the town both works for the factory and has cancer?

   (d) Are the events "this person works for the factory" and "this person has cancer" independent?

   (e) What is the probability that a random factory worker has cancer?

   (f) What is the probability that a random *resident who is not a factory worker* has cancer?

   (g) Last but not least, were either of the claims from the factory's public relations office false?

   Hint: it might be helpful to make a Venn Diagram to organize the data, with one circle being the people who have cancer, and one circle being the people who work for the factory.

Foreshadowing: Suppose we know that event $\mathcal{X}$ is independent of event $\mathcal{Y}$, and that we also know that event $\mathcal{Y}$ is independent of event $\mathcal{Z}$. Does this automatically indicate that event $\mathcal{X}$ is independent of event $\mathcal{Z}$? Think about it a little bit. We'll actually solve this explicitly as the last problem of this module.

5. Let us suppose that the probability of Alice being in class today is independent of the probability of Bob being in class today. (It is worth thinking about the reliability of this assumption.) If Alice attends with probability 95% and Bob attends with probability 85%, then what is the probability of both of them being present? both absent? one of them present and the other absent?

6. Imagine that you have an internship at a factory that manufactures smart watches. The suppliers of the components are fairly good, as it turns out. There are three components that might be defective: the accelerometer (to measure movement), the screen, and the battery. These fail with probability 0.005, 0.02, and 0.01. All other defects are extremely rare. Since the components come from completely different suppliers, scattered around the world, the probabilities of defects are definitely independent. What is the probability that a random watch will have no defective components? What is the probability of a random watch having at least one defect?

7. As we learned in the module "A Formal Introduction to Probability Theory," there is an approximately 2% chance that a rocket launch of a satellite will end in a complete disaster, such as the rocket exploding on the launch pad or the stages failing to separate and the rocket plummeting into the ocean. Of course, that also means there is a high probability of a disaster not happening at all. In any case, a new manager at a commercial satellite-launching firm has recently arrived, and has chided the launch operations department for the fact that they have never had 40 consecutive launches without an accident—their record for consecutive launches without an accident 38.

   To help the new manager understand matters, compute the probability of 39 consecutive launches without a single failure.

8. Suppose your friend Speedy is a very bad driver, and gets into a car accident with probability 1% on any given day. If he drives every day for one year, what is the probability that he gets into one or more car accidents during that year? Use a 365-day year.

9. Suppose a new website is booming. The company decides to get 8 servers, scattered around the world. Since the servers are around the world, independence is a fair assumption. Each server has a 99% up time. What is the probability that, at any given moment, there is at least one server that is down?

Background: At many computing conferences overseas, it is not uncommon to see prostitutes propositioning the attendees, particularly in the hotel bars in the evenings. This is most visible in the countries where it is perfectly legal (or in Nevada, where it is legal). Yet, it also occurs in countries or cities where it is not legal, but where the prohibition is not actively enforced (e.g. DC, New Orleans, or London). The idea is that computer scientists are often gullible, well-paid, and somewhat frustrated. Besides the legal risks, which might or might not exist depending on where one is located, there is the very real risk of disease—which we can now quantify using probability.

Back in 1998, I was chatting with a fellow who told me that he was paying his way through law school by being a male prostitute. Of course, I tried to convince him that this was a phenomenally unwise choice of part-time job. Again, besides being against the law, there is a risk of venereal disease. While he said that he was particularly careful to be safe, I specifically recall using the following probability argument to convince him to stop.

10. The male prostitute mentioned in the previous paragraph estimated that he had 3 sexual partners per week. Suppose the chance of getting a venereal disease on each encounter is 0.5%. Since law school is normally three years long, let's assume that he keeps up this pace for exactly three years. What is the probability that he would acquire at least one venereal disease during these three years?

    Before we can answer this question, we have to address independence. He felt as though there were extremely few repeat customers, and this would therefore allow us to use independence as a fairly safe assumption. If there were many repeat customers, then independence would be a poor model. Therefore, you could not use this model (at all) to describe a monogamous relationship, nor could you use it to model a business executive who has a wife and a mistress, but who is otherwise monogamous.

With all that in mind, compute for me now the probability that he would acquire at least one venereal disease during these three years?

Fun Story $10\frac{1}{2}$: A husband and wife that I was very good friends with have a lovely story about probability theory. She called me one day, unexpectedly, and told me that she had just taken a pregnancy test. Those are not very reliable, but it indicated that she was pregnant. She had sent her husband to the pharmacy, to buy a second pregnancy test kit, and meanwhile called me. I said "Oh, that's a great idea, then the probability of a wrong result goes from p to p^2." She laughed hysterically. Her husband and I went through our engineering degrees almost simultaneously at RPI (the Rensselaer Polytechnic Institute). We had taken the same probability course, "Modeling and Analysis of Uncertainty," though not at the same time. He had said exactly the same thing about p^2, almost word for word.

By the way, the fetus in question is now a happy and healthy teenager. In the end though, it is good to think about these things—if the probability of an inaccuracy is $p = 5\%$, then it becomes $p^2 = 0.05^2 = 0.0025 = 0.25\%$ instead.

11. Imagine a computing firm that manufactures ultra-durable tablet computers used on construction sites. The tablets have to be sturdy to keep up with the rigors of the job. An intern is told to look at some of the tablets that have been damaged but were still under warranty, and which accordingly got sent back to the company. This is the data that he observed:

    - The probability that the screen is damaged, and the motherboard is damaged $= 1/9$
    - The probability that the screen is damaged, and the motherboard is not damaged $= 2/9$
    - The probability that the screen is not damaged, and the motherboard is damaged $= 2/9$
    - The probability that the screen is not damaged, and the motherboard is not damaged $= 4/9$
    - The last one might seem surprising, but there are many components internal to a tablet PC, and perhaps components other than the screen or the motherboard are damaged.

    Let's now define the following events:

    - Let $\mathcal{A}$ be the event that the screen is damaged.
    - Let $\mathcal{B}$ be the event that the motherboard is damaged.
    - Let $\mathcal{C}$ be the event that the screen is not damaged.

    Based on those definitions, let's compute the following probabilities:

    (a) What is $Pr[\mathcal{A} \cup \mathcal{B}]$?
    (b) What is $Pr[\mathcal{A}]$?
    (c) What is $Pr[\mathcal{B}]$?
    (d) What is $Pr[\mathcal{C}]$?
    (e) What is $Pr[\mathcal{A} \cap \mathcal{B}]$?
    (f) What is $Pr[\mathcal{A}]Pr[\mathcal{B}]$?
    (g) Are $\mathcal{A}$ and $\mathcal{B}$ independent events?
    (h) What is $Pr[\mathcal{B} \cap \mathcal{C}]$?
    (i) What is $Pr[\mathcal{B}]Pr[\mathcal{C}]$?
    (j) Are $\mathcal{B}$ and $\mathcal{C}$ independent events?

(k) What is $Pr[\mathcal{A} \cap \mathcal{C}]$?

(l) What is $Pr[\mathcal{A}]Pr[\mathcal{C}]$?

(m) Are $\mathcal{A}$ and $\mathcal{C}$ independent events?

(n) In general, does the fact that $\mathcal{X}$ and $\mathcal{Y}$ are independent events, and the fact that $\mathcal{Y}$ and $\mathcal{Z}$ are independent events, guarantee that $\mathcal{X}$ and $\mathcal{Z}$ are independent events?

(o) Now that we've established that $\mathcal{A}$ and $\mathcal{C}$ are not independent events, what exactly is the relationship between them?

12. Suppose that in a particular Alaskan oil field, the chance of striking oil after drilling is 5%. If the drilling sites are sufficiently far apart, then the attempts are independent. The boss wants to report a successful oil strike this month, because the COO is coming to visit. He thinks that 20 drillings should guarantee a success. He thinks this because $(5\%)(20) = 100\%$.

    Of course, he's wrong. If there are 20 drillings, sufficiently separated geographically to guarantee independence, what is the probability of striking oil at least once?

13. Re-examine the previous problem. Using the repetition formulas, compute the number of drillings required, so that the chance of striking oil at least once is 95% or greater.

14. If events $\mathcal{A}$ and $\mathcal{B}$ are independent, then are the events $\mathcal{A}^c$ and $\mathcal{B}^c$ always independent as well? What do you think? (This is actually a fairly hard question.)

3 Answers

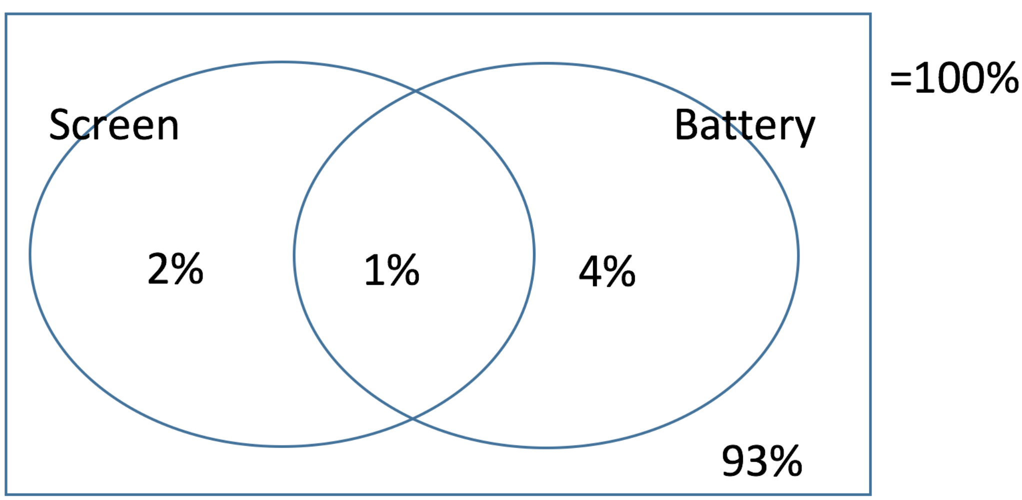
1. $Pr\{M\}Pr\{A\} = (0.6)(0.95) = 0.57$ and $Pr\{M \cap A\} = 0.57$ so, yes. The events are independent.

2. We will proceed by steps:
   - The probability of both flaws is $Pr\{S \cap B\} = Pr\{S\}Pr\{B\} = (0.03)(0.05) = 0.0015$ or 0.15%.
   - The probability of neither flaw is $Pr\{S^c \cap B^c\} = Pr\{S^c\}Pr\{B^c\} = (0.97)(0.95) = 0.9215$, or 92.15%.
   - The probability of refurbishment is $1 - 0.0015 - 0.9215 = 0.077$, or 7.70%.

3. The independence assumption was wrong, because

$$Pr\{S \cap B\} = Pr\{S\}Pr\{B\} = (0.03)(0.05) = 0.0015 \neq 0.01$$

but then we can still compute the probability distribution by making a Venn Diagram.

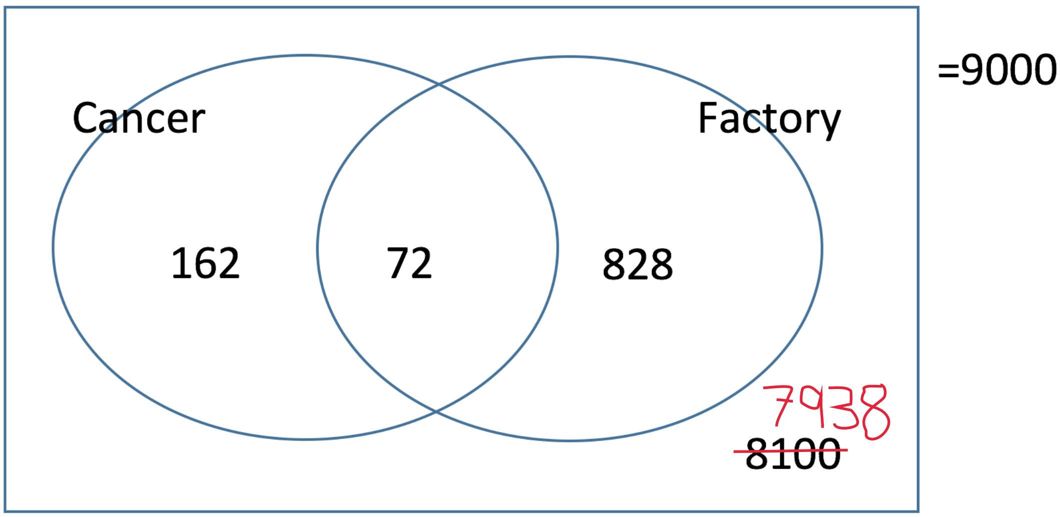


We conclude that 93% will be shipped out for sale, 6% will be refurbished, and 1% will be thrown out.

This non-independence can easily be imagined. For example, if the wave-soldering soldering machine is faulty, it might screw up both the battery and the screen. This is one influence affecting both variables simultaneously. That will make the intersection (the presence of both flaws) far more likely than one would otherwise expect. (In this case, more than $6\times$ more likely.)

4. Here are the answers to the question about the town with a cancer problem.

Hint: We begin by making the Venn Diagram.



(a) $Pr\{\text{Cancer}\} = \frac{234}{9000} = 0.026$.

(b) $Pr\{\text{Factory}\} = \frac{900}{9000} = 0.1$.

(c) $Pr\{\text{Cancer} \cap \text{Factory}\} = \frac{72}{9000} = 0.008$.

(d) First we compute,

$$Pr\{\text{Cancer}\}Pr\{\text{Factory}\} = (0.026)(0.1) = 0.0026$$

and of course

$$0.0026 \neq 0.008$$

which means that clearly, the events are not independent.

(e) The probability that a random factory worker has cancer is $\frac{72}{900} = 0.08$.

(f) The probability that a random person who is not a factory worker has cancer is $\frac{162}{162+7838} = \frac{162}{8100} = 0.02$.

Summary: As you can see, the probability of a person who works for the factory having cancer is *four times higher* than the probability of a person who does not work for the factory having cancer. This is obviously a serious problem. Moreover, the overall rate of cancer in the town is elevated, but still within the realm of possibility.

(g) Let's look now at those two claims from the public relations department:

- There are 234 people who have cancer. Two thirds of 234 is 156. There are 162 people who have cancer but who do not work at the factory. Since $162 > 156$, it is true that more than two-thirds of the cancer patients in the town are not employed at the factory. Adding the phrase "in any capacity whatsoever" is a common smoke screen. You bring undue emphasis on a minor irrelevant detail to distract from other aspects which might be embarrassing.

- Since 72 out of 900 employees of the factory have cancer, then 828 employees do not. We compute $828 \div 900 = 0.92 = 92\%$. This statistic is exactly true. Of course, for something potentially fatal like cancer, 92% is far too low. You'd like 98% or 99% of the town to be cancer-free, or better.

5. We have three things to compute.
   - Let A be the probability that Alice is present, and let B be the probability that Bob is present.
   - The probability of both present is $Pr\{A \cap B\} = Pr\{A\}Pr\{B\} = (0.95)(0.85) = 0.8075$.
   - The probability of both absent is $Pr\{A^c \cap B^c\} = Pr\{A^c\}Pr\{B^c\} = (0.05)(0.15) = 0.0075$.
   - The probability of one-absent/one-present is $1 - 0.8075 - 0.0075 = 0.185$.

6. A lot of students, without thinking, will simply compute

$$(0.005)(0.02)(0.01) = 10^{-6}$$

which is one-in-a-million. That's actually the probability that, by coincidence, a smartwatch has all three defects.

We do not want defective components—we want working components! Therefore, we compute the complements

$$1 - 0.005 = 0.995 \qquad 1 - 0.02 = 0.98 \qquad 1 - 0.01 = 0.99$$

and then multiply to obtain

$$(0.995)(0.98)(0.99) = 0.965349$$

which means that there is 96.5349% chance that a watch has no defective components. The complement of that

$$1 - 0.965349 = 0.034651 = 3.4651\%$$

is the probability of a random watch having at least one defect.

Note that it is wrong, but a useful approximation, to compute

$$0.005 + 0.02 + 0.01 = 0.035 = 3.5\%$$

7. The key to this problem is to realize that we are repeating a successful launch, not a failed launch. If $\mathcal{D}$ is the probability of a disaster, then we know $Pr\{\mathcal{D}\} = 0.02$. While this is true, that's not the event that we're repeating. We want to repeat $\mathcal{D}^c$, and surely

$$Pr\{\mathcal{D}^c\} = 1 - 0.02 = 0.98$$

With that in mind, the probability of 39 consecutive launches without a single failure is

$$p^n = (0.98)^{39} = 0.454796\cdots$$

which is clearly less than half. While 45.4796% is not a small probability by any means, we should not be surprised that this event did not occur.

In contrast, many students will write

$$p^n = (0.02)^{39} = 5.49755\cdots \times 10^{-67}$$

which is phenomenally small. As it turns out, this is the probability that all 39 launches have an unfortunate accident (such as exploding on the launch pad), and none of them survive the launch process.

To put this in perspective, there are about 1.8×10^{57} protons, neutrons, and electrons in the entire solar system. For comparison, 10^{67} is ten billion times as large as 10^{57}. There is no easy way for us to wrap our mind around 10^{-67}.

8. The key here is to realize that he has a 1% chance of getting into an accident each day, but that means he has a 99% chance of not getting into an accident each day. We're actually interested in the latter case. We compute

$$p^n = (0.99)^{365} = 0.0255179\cdots$$

is the probability that he will get into 0 car accidents in 365 consecutive days. The complement principle gives us

$$1 - 0.0255179\cdots = 0.974482\cdots$$

for the probability that he will get into at least one car accident per 365 consecutive days.

In case you are reading this after reading Module 5.2: "The Binomial Distribution Formula," note that we don't want to use the Binomial Distribution Formula, because you'd have to do a sum over

$$x \in \{1, 2, 3, 4, \ldots, 364, 365\}$$

and no one wants to evaluate that formula 365 times.

9. Each server is up with probability 99%, so the probability that all 8 of them are up is

$$p^n = (0.99)^8 = 0.922744\cdots$$

The complement principle tells us that the probability of having one or more down is

$$1 - 0.922744\cdots = 0.0772553\cdots$$

10. There will be $(52)(3) = 156$ partners per year. Thus, over the three years there will be $3(156) = 468$. Since the chance of acquiring a venereal disease on any particular encounter is 0.005, the chance of not acquiring one is 0.995. Then the chance of not acquiring one over 468 independent encounters is given by the repetition formula:

$$p^n = 0.995^{468} = 0.0957638\cdots$$

However, we were asked the probability of him acquiring at least one venereal disease, which is the complement of that.

$$1 - 0.0957638\cdots = 0.904236\cdots = 90.4236\%$$

In conclusion, there is a 90.4236% chance that he will acquire at least one venereal disease.

11. Here are the solutions to the question about the ultra-durable tablet PCs.

    (a) What is $Pr[\mathcal{A} \cup \mathcal{B}]$? [Answer: 5/9.]
    (b) What is $Pr[\mathcal{A}]$? [Answer: 1/3.]
    (c) What is $Pr[\mathcal{B}]$? [Answer: 1/3.]
    (d) What is $Pr[\mathcal{C}]$? [Answer: 2/3.]
    (e) What is $Pr[\mathcal{A} \cap \mathcal{B}]$? [Answer: 1/9.]
    (f) What is $Pr[\mathcal{A}]Pr[\mathcal{B}]$? [Answer: 1/9.]
    (g) Are $\mathcal{A}$ and $\mathcal{B}$ independent events? [Answer: Since $Pr[\mathcal{A} \cap \mathcal{B}] = 1/9 = Pr[\mathcal{A}]Pr[\mathcal{B}]$, we must say yes.]
    (h) What is $Pr[\mathcal{B} \cap \mathcal{C}]$? [Answer: 2/9.]
    (i) What is $Pr[\mathcal{B}]Pr[\mathcal{C}]$? [Answer: 2/9.]
    (j) Are $\mathcal{B}$ and $\mathcal{C}$ independent events? [Answer: Since $Pr[\mathcal{B} \cap \mathcal{C}] = 2/9 = Pr[\mathcal{B}]Pr[\mathcal{C}]$, we must say yes.]

(k) What is $Pr[\mathcal{A} \cap \mathcal{C}]$? [Answer: 0.]

(l) What is $Pr[\mathcal{A}]Pr[\mathcal{C}]$? [Answer: 2/9.]

(m) Are $\mathcal{A}$ and $\mathcal{C}$ independent events? [Answer: Since $Pr[\mathcal{A} \cap \mathcal{C}] = 0 \neq 2/9 = Pr[\mathcal{A}]Pr[\mathcal{C}]$, we must say no. The events $\mathcal{A}$ and $\mathcal{C}$ are not independent.]

(n) In general, does the fact that $\mathcal{X}$ and $\mathcal{Y}$ are independent events, and the fact that $\mathcal{Y}$ and $\mathcal{Z}$ are independent events, guarantee that $\mathcal{X}$ and $\mathcal{Z}$ are independent events? [Answer: Clearly not, as this problem provides us with an obvious counter-example.]

(o) Now that we've established that $\mathcal{A}$ and $\mathcal{C}$ are not independent events, what exactly is the relationship between them? [Answer: They are complements. To be precise $\mathcal{A}^c = \mathcal{C}$ and $\mathcal{C}^c = \mathcal{A}$.]

12. One drilling will fail to strike oil with probability 95%. Then all 20 of them will fail with probability

$$p^n = (0.95)^{20} = 0.358485\cdots$$

This means that striking oil at least once will occur with probability

$$1 - 0.358485\cdots = 0.641514\cdots$$

13. If the chance of striking oil at least once is 95% or greater, that means that the chance of missing oil each and every time is 5% or less. Each drilling has a 95% chance of failure. We have

$$\begin{aligned} 0.05 &> 0.95^n \\ \log 0.05 &> \log(0.95^n) \\ \log 0.05 &> n \log 0.95 \\ \frac{\log 0.05}{\log 0.95} &> n \\ 58.4039 &> n \end{aligned}$$

Therefore, we need 59 or more drillings. We should see that 58 will not be enough. Let's check

- With 58 drillings, we have $0.95^{58} = 0.0510468\cdots$.
- With 59 drillings, we have $0.95^{59} = 0.0484945\cdots$.
- This seems to be correct!

14. Yes. It turns out that if events $\mathcal{A}$ and $\mathcal{B}$ are independent, then the events $\mathcal{A}^c$ and $\mathcal{B}^c$ are independent also. Since this is true, I cannot just show you one example. I have to write a mathematical proof.

Claim: If events $\mathcal{A}$ and $\mathcal{B}$ are independent, then the events $\mathcal{A}^c$ and $\mathcal{B}^c$ are independent also.

Proof: Assume $\mathcal{A}$ and $\mathcal{B}$ are independent.

From DeMorgan's Laws, we know that

$$\mathcal{A}^c \cap \mathcal{B}^c = (\mathcal{A} \cup \mathcal{B})^c$$

which means that

$$Pr\{\mathcal{A}^c \cap \mathcal{B}^c\} = Pr\{(\mathcal{A} \cup \mathcal{B})^c\}$$

and by the complement principle

$$Pr\{\mathcal{A}^c \cap \mathcal{B}^c\} = Pr\{(\mathcal{A} \cup \mathcal{B})^c\} = 1 - Pr\{\mathcal{A} \cup \mathcal{B}\}$$

Now, recall the inclusion-exclusion principle:

$$Pr\{\mathcal{A} \cup \mathcal{B}\} = Pr\{\mathcal{A}\} + Pr\{\mathcal{B}\} - Pr\{\mathcal{A} \cap \mathcal{B}\}$$

Combining these, we have

$$Pr\{\mathcal{A}^c \cap \mathcal{B}^c\} = 1 - Pr\{(\mathcal{A} \cup \mathcal{B}\} = 1 - (Pr\{\mathcal{A}\} + Pr\{\mathcal{B}\} - Pr\{\mathcal{A} \cap \mathcal{B}\})$$

and we can distribute the minus sign to get

$$Pr\{\mathcal{A}^c \cap \mathcal{B}^c\} = 1 - Pr\{(\mathcal{A} \cup \mathcal{B}\} = 1 - Pr\{\mathcal{A}\} - Pr\{\mathcal{B}\} + Pr\{\mathcal{A} \cap \mathcal{B}\}$$

Since $\mathcal{A}$ and $\mathcal{B}$ are independent, we know that

$$Pr\{\mathcal{A} \cap \mathcal{B}\} = Pr\{\mathcal{A}\}Pr\{\mathcal{B}\}$$

so we can plug that in to obtain

$$\begin{aligned}
Pr\{\mathcal{A}^c \cap \mathcal{B}^c\} &= 1 - Pr\{\mathcal{A}\} - Pr\{\mathcal{B}\} + Pr\{\mathcal{A} \cap \mathcal{B}\} \\
Pr\{\mathcal{A}^c \cap \mathcal{B}^c\} &= 1 - Pr\{\mathcal{A}\} - Pr\{\mathcal{B}\} + Pr\{\mathcal{A}\}Pr\{\mathcal{B}\} \\
Pr\{\mathcal{A}^c \cap \mathcal{B}^c\} &= 1 - Pr\{\mathcal{A}\} - Pr\{\mathcal{B}\}\left(1 - Pr\{\mathcal{A}\}\right) \\
Pr\{\mathcal{A}^c \cap \mathcal{B}^c\} &= (1)\left(1 - Pr\{\mathcal{A}\}\right) - Pr\{\mathcal{B}\}\left(1 - Pr\{\mathcal{A}\}\right) \\
Pr\{\mathcal{A}^c \cap \mathcal{B}^c\} &= \left(1 - Pr\{\mathcal{B}\}\right)\left(1 - Pr\{\mathcal{A}\}\right) \\
Pr\{\mathcal{A}^c \cap \mathcal{B}^c\} &= Pr\{\mathcal{B}^c\}Pr\{\mathcal{A}^c\} \\
Pr\{\mathcal{A}^c \cap \mathcal{B}^c\} &= Pr\{\mathcal{A}^c\}Pr\{\mathcal{B}^c\}
\end{aligned}$$

That last line indicates that $\mathcal{A}^c$ and $\mathcal{B}^c$ are independent events.

In conclusion, if events $\mathcal{A}$ and $\mathcal{B}$ are independent, then the events $\mathcal{A}^c$ and $\mathcal{B}^c$ are independent also. ■