Module A-5: Injective, Surjective, and Bijective Functions

Math-270: Discrete Mathematics

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Motivation

You're surely familiar with the idea of an inverse function: a function that undoes some other function. For example,

$$f(x) = x^3 \qquad \text{and} \qquad g(x) = \sqrt[3]{x}$$

are inverses of each other. Whether thinking mathematically or coding this in software, things get complicated.

The theory of injective, surjective, and bijective functions is a very compact and mostly straightforward theory. Yet it completely untangles all the potential pitfalls of inverting a function.

Terminology

If a function f maps a set $\mathcal{X}$ to a set $\mathcal{Y}$, we are accustomed to calling $\mathcal{X}$ the domain (which is fine) but we are also accustomed to calling $\mathcal{Y}$ the range, and that is sloppy. The range of f is the set of values actually hit by f. In other words, y is in the range of $f(x)$ if and only if there is some x in the domain such that $f(x) = y$. Without this restriction, we refer to $\mathcal{Y}$ as the co-domain of $f(x)$.

You have probably heard the phrase "y is the image of x" when $f(x) = y$. Likewise, we can say "x is a pre-image of y." Notice that we say "a pre-image" and not "the pre-image." That's because y might have multiple pre-images.

For example, if $f(x) = x^2$ as a function of the real line, then $y = 4$ has two pre-images: $x = 2$ and $x = -2$. Meanwhile, $y = 0$ has only one pre-image, $x = 0$. In contrast, $y = -1$ has no pre-images.

Injective Functions

Formal Defintion: A function $f : D \to C$ is injective if and only if

"for all $x_1 \in D$ and $x_2 \in D$ if $f(x_1) = f(x_2)$ then $x_1 = x_2$."

Casual Definition: No two distinct points in the domain map to the same value.

Classic Example: $f(x) = e^x$, thought of as $f : \mathbb{R} \to \mathbb{R}$.

Horizontal Line Test: Every horizontal line hits the curve *at most once.*

Easy Non-Example: $f(x) = \sin x$, thought of as $f : \mathbb{R} \to \mathbb{R}$.

Pre-images: Every point in the co-domain has at most one pre-image.

Surjective Functions

Formal Defintion: A function $f : D \to C$ is surjective if and only if

"for all $y \in C$ there exists an $x \in D$ such that $f(x) = y$."

Casual Definition: Every point in the co-domain has some point in the domain that maps to it.

Classic Example: $f(x) = \tan x$, thought of as $\mathbb{R} - \left\{\ldots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\right\} \to \mathbb{R}$.

Horizontal Line Test: Every horizontal line hits the curve *at least once.*

Easy Non-Example: $f(x) = e^x$, thought of as $f : \mathbb{R} \to \mathbb{R}$.

Pre-images: Every point in the co-domain has at least one pre-image.

Bijective Functions

Formal Defintion: A function f is bijective if and only if it is both injective and surjective.

Casual Definition: Every point in the co-domain has exactly one point in the domain that maps to it.

Classic Example: $f(x) = x^3$, thought of as $\mathbb{R} \to \mathbb{R}$.

Horizontal Line Test: Every horizontal line hits the curve *exactly once.*

Easy Non-Example: $f(x) = x^2$, thought of as $f : \mathbb{R} \to \mathbb{R}$.

Pre-images: Every point in the co-domain has exactly one pre-image.

Mathematical Terminology Used in Other Textbooks

- Injective functions are sometimes called "injections," which is fine.
- Surjective functions are sometimes called "surjections," which is fine.
- Bijective functions are often called "bijections," which is fine.
- In 100-level courses, we sometimes say "$f(x)$ is invertible" instead of "$f(x)$ is bijective," and that's okay. It would traumatize 100-level students to learn about these finer distinctions which (at that level) would be seriously confusing.
- In some textbooks, injective functions are called "one-to-one" functions, especially at lower levels. However, this phrase is sometimes used for bijections, and therefore, it should be avoided. When you see the phrase "one-to-one functions," it is ambiguous, because some authors will use that phrase to indicate injective functions, and some will use that phrase to indicate bijective functions. As always, the best plan is to avoid ambiguity entirely, and use the formal vocabulary instead of mathematical slang.
- In some textbooks, we might see "the function $f(x)$ is onto" in place of "the function $f(x)$ is surjective." However, this is painful to an ear accustomed to proper grammar, and should not be used. Also the phrases "$f(x)$ maps $\mathcal{A}$ onto $\mathcal{B}$" versus "$f(x)$ maps $\mathcal{A}$ into $\mathcal{B}$" are too similar to each other. The human ear might mistake one for the other, but the former indicates a surjective function, whereas the latter does not say anything about surjectivity.

The Three Formal Definitions

Here is a recap of the formal definitions, for ease of reference.

- A function $f : D \to C$ is injective if and only if

  "for all $x_1 \in D$ and $x_2 \in D$ if $f(x_1) = f(x_2)$ then $x_1 = x_2$."

- A function $f : D \to C$ is surjective if and only if
  "for all $y \in C$ there exists an $x \in D$ such that $f(x) = y$."
- A function f is bijective if and only if it is both injective and surjective.

How to Prove These Things

Surjective

The best way to prove that some function is surjective is to provide a formula that, given any y-value in the co-domain, will produce an x-value in the domain such that $f(x) = y$.

In other words, if someone asks you to prove that something exists, the best way to accomplish this is to produce that very thing.

The negation of the formal definition is fairly interesting.

$$\begin{array}{l} \sim (\forall y \in C \quad \exists x \in D \quad f(x) = y) \\ \exists y \in C \quad \sim (\exists x \in D \quad f(x) = y) \\ \exists y \in C \quad \forall x \in D \quad \sim (f(x) = y) \\ \exists y \in C \quad \forall x \in D \quad f(x) \neq y \end{array}$$

As you can see, the recipe (for proving that a function is *not* surjective) is to locate some y-value in the co-domain, for which there is no x-value in the domain where $f(x) = y$.

Injective

The best way to prove that some function is injective is to use a direct proof.

Let $x_1 \in D$, let $x_2 \in D$, and suppose that $f(x_1) = f(x_2)$. Then you do some algebra, until you reach $x_1 = x_2$. Conclude "if $f(x_1) = f(x_2)$ then $x_1 = x_2$, therefore f is injective."

Again, the negation of the formal definition is very interesting.

$$\begin{array}{l} \sim (\forall x_1 \in D \quad \forall x_2 \in D \quad \text{if } f(x_1) = f(x_2) \text{ then } x_1 = x_2) \\ \exists x_1 \in D \quad \sim (\forall x_2 \in D \quad \text{if } f(x_1) = f(x_2) \text{ then } x_1 = x_2) \\ \exists x_1 \in D \quad \exists x_2 \in D \quad \sim (\text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2) \\ \exists x_1 \in D \quad \exists x_2 \in D \quad [(f(x_1) = f(x_2)) \text{ and } \sim (x_1 = x_2)] \\ \exists x_1 \in D \quad \exists x_2 \in D \quad [(f(x_1) = f(x_2)) \text{ and } x_1 \neq x_2] \end{array}$$

As you can see, the way to prove that some function is *not* injective is to produce some $x_1 \in D$ and some $x_2 \in D$ with $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. A single pair of values meeting these requirements will be enough. Often, this is very easy to do.

Bijective

The principle way to prove that something is bijective is to show (in two separate proofs) that it is injective and surjective. This can take up a lot of space.

There are other options. For example, the composition of two or more bijections is a bijection. So if you knew that $a(x) = \sqrt[3]{x}$, $b(x) = x + 1$ and $c(x) = x^5$ are all bijections from $\mathbb{R} \to \mathbb{R}$ then you can know immediately that

$$f(x) = \sqrt[3]{x^5 + 1}$$

is bijective, because

$$f(x) = a(b(c(x)))$$

To show that a function is not bijective, the only way that I know is to show either that it fails to be injective, or that it fails to be surjective. You would use the recipes noted above, as appropriate.

Exercises

My favorite examination technique for this topic is for the student to be challenged with 2–4 functions. For each one, the student will be asked if the function is injective, if the function is surjective, and if the function is bijective. This way, it will be a question that can be rapidly answered, and rapidly graded. Nonetheless, as you study, try to think in the formal language of proof writing, to avoid making errors.

1. $f(x) = x^2$, thought of as $\mathbb{R} \to \mathbb{R}$.
2. $f(x) = x^2$, thought of as "positive reals" $\to$ "positive reals."
3. $f(x) = |x|$, thought of as $\mathbb{R} \to \mathbb{R}$.
4. $f(x) = \sqrt{x}$, thought of as "positive reals" $\to$ "positive reals."
5. $f(x) = e^x$, thought of as $\mathbb{R} \to \mathbb{R}$.
6. $f(x) = \ln x$, thought of as "positive reals" $\to \mathbb{R}$.
7. $f(x) = \sqrt[3]{x}$, thought of as $\mathbb{R} \to \mathbb{R}$.
8. $f(x) = 2^x$, thought of as $\mathbb{R} \to \mathbb{R}$.
9. $f(x) = \sin x$, thought of as $\mathbb{R} \to \mathbb{R}$.
10. $f(x) = \sin x$, thought of as having a domain of $0 \le x < 2\pi$ and a co-domain of $-1 \le y \le 1$.
11. $f(x) = \cos x$, thought of as $\mathbb{R} \to \mathbb{R}$.
12. $f(x) = \cos x$, thought of as having a domain of $0 \le x < 2\pi$ and a co-domain of $-1 \le y \le 1$.
13. $f(x) = \tan x$, thought of as $\mathbb{R} - \left\{\ldots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\right\} \to \mathbb{R}$.
14. $f(x) = x^3$, thought of as $\mathbb{R} \to \mathbb{R}$.
15. $f(x) = x^5$, thought of as $\mathbb{R} \to \mathbb{R}$.
16. $f(x) = 3x + 8$, thought of as $\mathbb{R} \to \mathbb{R}$.
17. $f(x) = x$, thought of as $\mathbb{R} \to \mathbb{R}$.

Note: For the next one, recall that for a complex number $a + bi$, the complex conjugate is $\overline{a + bi} = a - bi$.

18. $f(z) = z \cdot \overline{z}$, thought of as having a domain of $\mathbb{C} \to \mathbb{C}$.
19. $f(x) = \frac{x+5}{x-4}$, thought of as having a domain of all real numbers not equal to 4, and a co-domain of all real numbers.

Note: The solutions begin on the next page of this file.

Solutions

1. Fails to be injective, fails to be surjective, fails to be bijective.
2. Is injective, is surjective, is bijective.
3. Fails to be injective, fails to be surjective, fails to be bijective.
4. Is injective, is surjective, is bijective.
5. Is injective, fails to be surjective, fails to be bijective.
6. Is injective, is surjective, is bijective.
7. Is injective, is surjective, is bijective.
8. Is injective, fails to be surjective, fails to be bijective.
9. Fails to be injective, fails to be surjective, fails to be bijective.
10. Fails to be injective, is surjective, fails to be bijective.
11. Fails to be injective, fails to be surjective, fails to be bijective.
12. Fails to be injective, is surjective, fails to be bijective.
13. Fails to be injective, is surjective, fails to be bijective.
14. Is injective, is surjective, is bijective.
15. Is injective, is surjective, is bijective.
16. Is injective, is surjective, is bijective.
17. Is injective, is surjective, is bijective.
18. Fails to be injective, fails to be surjective, fails to be bijective.
19. Is injective, fails to be surjective ($y = 1$), fails to be bijective.