Quick Reference

- We say "R is a relation from $\mathcal{S}$ to $\mathcal{T}$" if and only if R is a subset of $\mathcal{S} \times \mathcal{T}$.
- By $\times$ we mean the Cartesian Product.
- We say "R is a relation on $\mathcal{S}$" if and only if R is a subset of $\mathcal{S} \times \mathcal{S}$.
- We will write "aRb is true" if and only if $(a, b) \in R$.
- We will write "aRb is false" if and only if $(a, b) \notin R$.
- In computer science terms, R is a function that takes two inputs, both of type $\mathcal{S}$, and returns a boolean—either true or false.
- We say that "a relation R on $\mathcal{S}$ is reflexive" if and only if for all $a \in \mathcal{S}$, aRa is true.
- We say that "a relation R on $\mathcal{S}$ is symmetric" if and only if for all $a \in \mathcal{S}$ and $b \in \mathcal{S}$, if aRb is true, then bRa is true.
- We say that "a relation R on $\mathcal{S}$ is transitive" if and only if for all $a, b, c \in \mathcal{S}$, if both aRb is true and bRc is true, then aRc is true.
- We say that "a relation R is an equivalence relation" if and only if R is reflexive, symmetric, and transitive.
- If a relation R on $\mathcal{S}$ is an equivalence relation, then it partitions $\mathcal{S}$ into sets, called "equivalence classes" such that (1) the equivalence classes are mutually exclusive, (2) the equivalence classes are collectively exhaustive, and (3) each class C has aRb being true for all $a, b \in C$.

Questions

For each of the listed situations, I want you to figure out the following:

(a) Is this relation reflexive?

(b) Is this relation symmetric?

(c) Is this relation transitive?

(d) Is this relation an equivalence relation?

(e) If your answer to (d) was yes, then try to describe the equivalence classes.

Here are the situations...

Note: In every case, R is a relation on $\mathcal{S}$.

1. Let $\mathcal{S}$ be the set of courses at some university. Let aRb if and only if a is a prerequisite for b.
2. Let $\mathcal{S}$ be the set of people with driver's licenses in Wisconsin. Let aRb if and only if a is taller than b.
3. Let $\mathcal{S}$ be the set of people with driver's licenses in Wisconsin. Let aRb if and only if a is *not* taller than b.
4. Let $\mathcal{S}$ be the set of people with driver's licenses in Wisconsin. Let aRb if and only if a and b have the same last name.
5. Let $\mathcal{S} = \mathbb{R}$. Let aRb if and only if $a \geq b$.

6. Let $\mathcal{S} = \mathbb{Q}$. Let aRb if and only if $a = b$.

7. Let $\mathcal{S} = \mathbb{Z}$. Let aRb if and only if $a^2 = b^2$.

8. Let $\mathcal{S} = \mathbb{R}$. Let aRb if and only if $\sin a = \sin b$.

9. Suppose you are studying a particular function $f : \mathbb{R} \to \mathbb{R}$. Let $\mathcal{S} = \mathbb{R}$. Let aRb if and only if $f(a) = f(b)$.

Note: The previous example (#9) is a generalization of the three before it (#6, #7, and #8). The functions would be $f(x) = x$, $f(x) = x^2$, and $f(x) = \sin x$.

10. Let $\mathcal{S} = \mathbb{Z}$. Let aRb if and only if a and b have a gcd greater than 1.

11. Let $\mathcal{S}$ be the positive integers. Let aRb if and only if a and b have the same number of divisors.

12. Let $\mathcal{S}$ be any set. Consider the enthusiastic relation. It always reports "true" when asked aRb about any $a \in \mathcal{S}$ and any $b \in \mathcal{S}$.

13. Let $\mathcal{S}$ be any set. Consider the pessimistic relation. It always reports "false" when asked aRb about any $a \in \mathcal{S}$ and any $b \in \mathcal{S}$.

14. Let $\mathcal{S} = \mathcal{P}(\mathcal{A})$, the power set of $\mathcal{A}$, where $\mathcal{A}$ can be any set. More plainly, $\mathcal{S}$ is the set of all subsets of $\mathcal{A}$. It always reports "true" when asked aRb about any $a \in \mathcal{S}$ and any $b \in \mathcal{S}$ if and only if $a \subseteq b$.

15. Let $\mathcal{S} = \mathcal{P}(\mathcal{A})$, the power set of $\mathcal{A}$, where $\mathcal{A}$ can be any finite set. More plainly, $\mathcal{S}$ is the set of all subsets of $\mathcal{A}$. It always reports "true" when asked aRb about any $a \in \mathcal{S}$ and any $b \in \mathcal{S}$ if and only if a and b are the same size.

16. Let $\mathcal{D} = \{1, 2, 3, 4, 5, 6\}$. Let $\mathcal{S} = \mathcal{D} \times \mathcal{D}$. In other words, $\mathcal{D}$ represents rolling one six-sided die, and $\mathcal{S}$ represents rolling two six-sided dice, and getting perhaps $(1, 3)$ or $(4, 6)$, and so forth. When asked aRb about any $a \in \mathcal{S}$ and any $b \in \mathcal{S}$, we will report true if and only if a and b have the same sum. This is like playing Battletech, Settlers of Catan, or Monopoly, where you really interpret $(1, 3)$ and $(2, 2)$ both as "4" and $(4, 6)$ and $(5, 5)$ both as "10." However, this model would fail for Backgammon, where rolling "doubles" (e.g. (2,2) or (5,5)) is special.

17. Let $\mathcal{S}$ be any set, that has been partitioned into $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \ldots$. That means that the $\mathcal{S}_i$s are mutually exclusive and collectively exhaustive. When asked aRb about any $a \in \mathcal{S}$ and any $b \in \mathcal{S}$, we will report true if a and b are in the same $\mathcal{S}_i$, and false if they are in different $\mathcal{S}_i$s.

Note: The previous problem is just a generalization of the one before it. When rolling two six-sided dice, the outcomes are things like $(2, 2)$, $(1, 3)$, $(4, 6)$, and $(5, 5)$. However, in most games, we really do treat $(2, 2)$ and $(1, 3)$ as being just 4, and likewise $(4, 6)$ and $(5, 5)$ are just 10. So it makes sense that we should organize the simple events into compound events, and make a probability distribution for the compound events. It is noteworthy, however, that $\mathcal{S}$ has the property that all simple events are equiprobable. With the equivalence classes, this is absolutely false. Anyone who plays Settlers of Catan, or Craps, knows that rolling a 7 is much more common than rolling a 2 or a 12. The probability of any such roll is the number of members of the equivalence class, divided by 36. In other words, because

$$\#\mathcal{S}_7 = \#\left\{(1, 6);\ (2, 5);\ (3, 4);\ (4, 3);\ (5, 2);\ (6, 1)\right\} = 6$$

the probability of rolling a 7 is $6/36 = 1/6$, and because

$$\#\mathcal{S}_{12} = \#\left\{(6, 6)\right\} = 1$$

the probability of rolling a 12 is $1/36$. Why 36? That's because $\mathcal{S}$ has $36 = 6^2$ members.

18. Let $\mathcal{S} = \mathbb{Z} \times \mathbb{Z}^\times$. Recall, $\mathbb{Z}^\times$ means the non-zero integers. So that means $\mathcal{S}$ consists of ordered pairs (x, y) such that x and y are both integers, and $y \neq 0$. We shall define $(x_1, y_1)R(x_2, y_2)$ if and only if $x_1 y_2 = x_2 y_1$.

19. Let $\mathcal{S} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. We shall say that

$$(x_1, y_1, z_1)R(x_2, y_2, z_2) \text{ if and only if } \exists k \in \mathbb{R}^\times \text{ such that } (x_2, y_2, z_2) = (kx_1, ky_1, kz_1)$$

but for this one, don't think about the equivalences classes, because your head might explode.

20. Let $\mathcal{S} = \mathbb{Z}$. Consider aRb if and only if a and b have the same parity. That means aRb is true if either a and b are both odd or both even; aRb is false if one is odd and the other is even.

21. Let $\mathcal{S} = \mathbb{Z}$. Consider aRb if and only if $a - b$ is a multiple of 3. Recall that

$$\text{multiples}(3) = \{\ldots, -15, -12, -9, -6, -3, 0, 3, 6, 9, 12, 15, \ldots\}$$

22. Let $\mathcal{S} = \mathbb{Z}$. Consider aRb if and only if $a - b$ is a multiple of 5. Recall that

$$\text{multiples}(5) = \{\ldots, -25, -20, -15, -10, -5, 0, 5, 10, 15, 20, 25, \ldots\}$$

Note: For the next four, remember: we only consider finite graphs in Math-270, not infinite graphs.

23. Let $\mathcal{S}$ be the set of vertices of some graph. Consider the relation "is adjacent to." In other words $v_1 R v_2$ if and only if v_1 is adjacent to v_2.

24. Let $\mathcal{S}$ be the set of vertices of some graph. Consider the relation "is connected to." In other words $v_1 R v_2$ if and only if $v_1 = v_2$ or if there exists a path from v_1 to v_2.

25. Let $\mathcal{S}$ be a collection of graphs. Consider $G_1 R G_2$ if and only if G_1 is isomorphic to G_2.

26. Let $\mathcal{S}$ be a collection of graphs. Consider $G_1 R G_2$ if and only if G_1 has the same number of vertices as G_2.

Note: The next five have to do with parentage. Imagine that there is a database of all birth certificates awarded in the USA in the last 200 years. The set $\mathcal{S}$ is the set of people issued birth certificates.

27. Let aRb if and only if a is the biological father of b.

28. Let aRb if and only if a and b have both biological parents in common.

29. Let aRb if and only if a and b have one biological parent in common.

30. Let aRb if and only if a is a biological ancestor of b.

31. Let aRb if and only if a and b have the same biological mother.

32. Let $\mathcal{S}$ be any set. Let aR_1b be an equivalence relation on $\mathcal{S}$. Define aR_2b to be the negation of aR_1b. In other words, aR_2b is true if and only if aR_1b is false.

33. Let $\mathcal{S}$ be any set. Let aR_1b and aR_2b be two equivalence relations on $\mathcal{S}$. Define aR_3b to be the union of aR_1b and aR_2b. In other words

$$(aR_3b) \Leftrightarrow [(aR_1b) \vee (aR_2b)]$$

34. Let $\mathcal{S}$ be any set. Let aR_1b and aR_2b be two equivalence relations on $\mathcal{S}$. Define aR_3b to be the intersection of aR_1b and aR_2b. In other words

$$(aR_3b) \Leftrightarrow [(aR_1b) \wedge (aR_2b)]$$

Answers

1. Let $\mathcal{S}$ be the set of courses at some university. Let aRb if and only if a is a prerequisite for b.
   (a) Reflexive: No
   (b) Symmetric: No
   (c) Transitive: Yes
   (d) Equivalence Relation: No
   (e) Equivalence Classes: n/a
2. Let $\mathcal{S}$ be the set of people with driver's licenses in Wisconsin. Let aRb if and only if a is taller than b.
   (a) Reflexive: No
   (b) Symmetric: No
   (c) Transitive: Yes
   (d) Equivalence Relation: No
   (e) Equivalence Classes: n/a
3. Let $\mathcal{S}$ be the set of people with driver's licenses in Wisconsin. Let aRb if and only if a is *not* taller than b.
   (a) Reflexive: Yes
   (b) Symmetric: No
   (c) Transitive: Yes
   (d) Equivalence Relation: No
   (e) Equivalence Classes: n/a
4. Let $\mathcal{S}$ be the set of people with driver's licenses in Wisconsin. Let aRb if and only if a and b have the same last name.
   (a) Reflexive: Yes
   (b) Symmetric: Yes
   (c) Transitive: Yes
   (d) Equivalence Relation: Yes
   (e) Equivalence Classes: There is one equivalence class for each last name, and it is the set of people who have that last name and who have Wisconsin driver's licenses.
5. Let $\mathcal{S} = \mathbb{R}$. Let aRb if and only if $a \geq b$.
   (a) Reflexive: Yes
   (b) Symmetric: No
   (c) Transitive: Yes
   (d) Equivalence Relation: No
   (e) Equivalence Classes: n/a
6. Let $\mathcal{S} = \mathbb{Q}$. Let aRb if and only if $a = b$.
   (a) Reflexive: Yes
   (b) Symmetric: Yes

(c) Transitive: Yes
(d) Equivalence Relation: Yes
(e) Equivalence Classes: A set of singleton sets, one for each rational number, consisting of only that rational number (alone by itself).

7. Let $\mathcal{S} = \mathbb{Z}$. Let aRb if and only if $a^2 = b^2$.

(a) Reflexive: Yes
(b) Symmetric: Yes
(c) Transitive: Yes
(d) Equivalence Relation: Yes
(e) Equivalence Classes: $\{\{0\}\,;\ \{-1,1\}\,;\ \{-2,2\}\,;\ \{-3,3\}\,;\ \{-4,4\}\,;\ \{-5,5\}\,;\ \ldots\}$

8. Let $\mathcal{S} = \mathbb{R}$. Let aRb if and only if $\sin a = \sin b$.

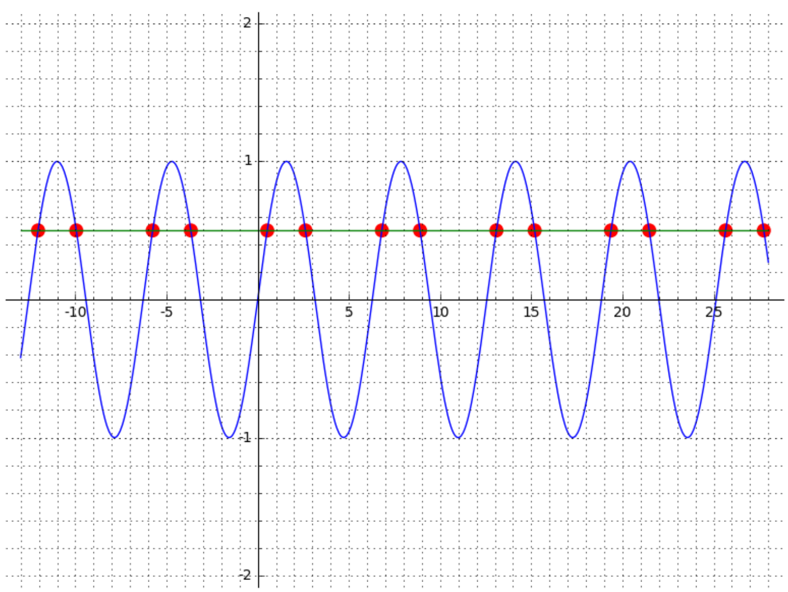
(a) Reflexive: Yes
(b) Symmetric: Yes
(c) Transitive: Yes
(d) Equivalence Relation: Yes
(e) Equivalence Classes: A set of singleton sets, each comprising of numbers that all have the same sine. For example, one equivalence class is

$$\left\{\ldots, \frac{-23\pi}{6}, \frac{-19\pi}{6}, \frac{-11\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \frac{37\pi}{6}, \frac{41\pi}{6}, \frac{49\pi}{6}, \frac{53\pi}{6}, \ldots\right\}$$

since all of those have a sine of $1/2$. However, I like to think in degrees, so I would write

$$\{\ldots, -690^\circ, -570^\circ, -330^\circ, -210^\circ, 30^\circ, 150^\circ, 390^\circ, 510^\circ, 750^\circ, 870^\circ, 1110^\circ, 1230^\circ, \ldots\}$$

Note: Another way to think about this is to consider all possible y coordinates, $-1 < y < 1$. There is one equivalence class for each y-coordinate. It has inside it all possible θ such that the triangle in the unit circle with that θ will have the required y coordinate (for the vertex of the triangle on the unit circle). Last but not least, perhaps a graph would help. (The code for making this graph is given on the last page of this workbook.)



9. Suppose you are studying a particular function $f : \mathbb{R} \to \mathbb{R}$. Let $\mathcal{S} = \mathbb{R}$. Let aRb if and only if $f(a) = f(b)$.
   (a) Reflexive: Yes
   (b) Symmetric: Yes
   (c) Transitive: Yes
   (d) Equivalence Relation: Yes
   (e) Equivalence Classes: There is an equivalence class for each possible y in the range of f. The equivalence class for y contains all the $x \in S$ such that $f(x) = y$. We say that this equivalence class is "the pre-image of y."

Note: The previous example (#9) is a generalization of the three before it (#7, #8, and #9). The functions would be $f(x) = x$, $f(x) = x^2$, and $f(x) = \sin x$.

10. Let $\mathcal{S} = \mathbb{Z}$. Let aRb if and only if a and b have a gcd greater than 1.
    (a) Reflexive: Yes
    (b) Symmetric: Yes
    (c) Transitive: No. Consider $a = 3$, $b = 6$, and $c = 2$.
    (d) Equivalence Relation: No
    (e) Equivalence Classes: n/a

11. Let $\mathcal{S}$ be the positive integers. Let aRb if and only if a and b have the same number of divisors.
    (a) Reflexive: Yes
    (b) Symmetric: Yes
    (c) Transitive: Yes
    (d) Equivalence Relation: Yes
    (e) Equivalence Classes: There is one equivalence class for each integer n. It consists of all positive integers such that divisors(z), as a set, has n elements. For example, all the primes are together in one class, because they all have two divisors, $\{1, p\}$. The squares of all the primes are together in one class, because they all have three divisors, $\left\{1, p, p^2\right\}$. All numbers that are the product of two distinct primes are together in one class, because they have four divisors, $\{1, p, q, pq\}$. *Et cetera...*

12. Let $\mathcal{S}$ be any set. Consider the enthusiastic relation. It always reports "true" when asked aRb about any $a \in \mathcal{S}$ and any $b \in \mathcal{S}$.
    (a) Reflexive: Yes
    (b) Symmetric: Yes
    (c) Transitive: Yes
    (d) Equivalence Relation: Yes
    (e) Equivalence Classes: One big class containing all of $\mathcal{S}$.

13. Let $\mathcal{S}$ be any set. Consider the pessimistic relation. It always reports "false" when asked aRb about any $a \in \mathcal{S}$ and any $b \in \mathcal{S}$.
    (a) Reflexive: No
    (b) Symmetric: Yes
    (c) Transitive: Yes

(d) Equivalence Relation: No

(e) Equivalence Classes: n/a

Hint: Why is the pessimistic relation symmetric? To disprove it being symmetric, you'd need a situation where aRb is true, but bRa is false. Since aRb is never true, you cannot possibly find such an a and b.

14. Let $\mathcal{S} = \mathcal{P}(\mathcal{A})$, the power set of $\mathcal{A}$, where $\mathcal{A}$ can be any set. More plainly, $\mathcal{S}$ is the set of all subsets of $\mathcal{A}$. It always reports "true" when asked aRb about any $a \in \mathcal{S}$ and any $b \in \mathcal{S}$ if and only if $a \subseteq b$.

    (a) Reflexive: Yes

    (b) Symmetric: No

    (c) Transitive: Yes

    (d) Equivalence Relation: No

    (e) Equivalence Classes: n/a

15. Let $\mathcal{S} = \mathcal{P}(\mathcal{A})$, the power set of $\mathcal{A}$, where $\mathcal{A}$ can be any finite set. More plainly, $\mathcal{S}$ is the set of all subsets of $\mathcal{A}$. It always reports "true" when asked aRb about any $a \in \mathcal{S}$ and any $b \in \mathcal{S}$ if and only if a and b are the same size.

    (a) Reflexive: Yes

    (b) Symmetric: Yes

    (c) Transitive: Yes

    (d) Equivalence Relation: Yes

    (e) Equivalence Classes: There is one equivalence class for each possible size. So if $\#\mathcal{A} = n$ then the possible sizes are $\{0, 1, 2, 3, 4, \ldots, n-1, n\}$. The equivalence class for some $x \in \{0, 1, 2, \ldots, n\}$ contains all subsets of $\mathcal{A}$ that have size x.

    Note: Cool property. Suppose that $\mathcal{A}$ has five elements. Of course, $\mathcal{S} = \mathcal{P}(\mathcal{A})$ has $2^5 = 32$ elements. How big are the equivalence classes? Well, there is 1 subset of size 0; 5 subsets of size 1; 10 subsets of size 2; 10 subsets of size 3; 5 subsets of size 4; and 1 subset of size 5. We get "1, 5, 10, 10, 5, 1" and that's the 5th row of Pascal's Triangle.

16. Let $\mathcal{D} = \{1, 2, 3, 4, 5, 6\}$. Let $\mathcal{S} = \mathcal{D} \times \mathcal{D}$. In other words, $\mathcal{D}$ represents rolling one six-sided die, and $\mathcal{S}$ represents rolling two six-sided dice, and getting perhaps $(1, 3)$ or $(4, 6)$, and so forth. When asked aRb about any $a \in \mathcal{S}$ and any $b \in \mathcal{S}$, we will report true if and only if a and b have the same sum. This is like playing Battletech, Settlers of Catan, or Monopoly, where you really interpret $(1, 3)$ and $(2, 2)$ both as "4" and $(4, 6)$ and $(5, 5)$ both as "10." However, this model would fail for Backgammon, where rolling "doubles" (e.g. (2,2) or (5,5)) is special.

    (a) Reflexive: Yes

    (b) Symmetric: Yes

    (c) Transitive: Yes

    (d) Equivalence Relation: Yes

    (e) Equivalence Classes: There is one equivalence class for each possible sum, $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and any particular equivalence class contains the different ways of rolling that sum.

17. Let $\mathcal{S}$ be any set, that has been partitioned into $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \ldots$. That means that the $\mathcal{S}_i$s are mutually exclusive and collectively exhaustive. When asked aRb about any $a \in \mathcal{S}$ and any $b \in \mathcal{S}$, we will report true if a and b are in the same $\mathcal{S}_i$, and false if they are in different $\mathcal{S}_i$s.

    (a) Reflexive: Yes

(b) Symmetric: Yes

(c) Transitive: Yes

(d) Equivalence Relation: Yes

(e) Equivalence Classes: The equivalence classes are just the $\mathcal{S}_i$s.

Note: The previous problem is just a generalization of the one before it. When rolling two six-sided dice, the outcomes are things like $(2,2)$, $(1,3)$, $(4,6)$, and $(5,5)$. However, in most games, we really do treat $(2,2)$ and $(1,3)$ as being just 4, and likewise $(4,6)$ and $(5,5)$ are just 10. So it makes sense that we should organize the simple events into compound events, and make a probability distribution for the compound events. It is noteworthy, however, that $\mathcal{S}$ has the property that all simple events are equiprobable. With the equivalence classes, this is absolutely false. Anyone who plays Settlers of Catan, or Craps, knows that rolling a 7 is much more common than rolling a 2 or a 12. The probability of any such roll is the number of members of the equivalence class, divided by 36. In other words, because

$$\#\mathcal{S}_7 = \#\left\{(1,6);\ (2,5);\ (3,4);\ (4,3);\ (5,2);\ (6,1)\right\} = 6$$

the probability of rolling a 7 is $6/36 = 1/6$, and because

$$\#\mathcal{S}_{12} = \#\left\{(6,6)\right\} = 1$$

the probability of rolling a 12 is $1/36$. Why 36? That's because $\mathcal{S}$ has $36 = 6^2$ members.

18. Let $\mathcal{S} = \mathbb{Z} \times \mathbb{Z}^\times$. Recall, $\mathbb{Z}^\times$ means the non-zero integers. So that means $\mathcal{S}$ consists of ordered pairs (x, y) such that x and y are both integers, and $y \neq 0$. We shall define $(x_1, y_1)R(x_2, y_2)$ if and only if $x_1y_2 = x_2y_1$.

(a) Reflexive: Yes

(b) Symmetric: Yes

(c) Transitive: Yes

(d) Equivalence Relation: Yes

Note: Before we describe the equivalence classes, one way to think of (x_1, y_1) is as the fraction x_1/y_1. Then we have $(x_1, y_1)R(x_2, y_2)$ if and only if $x_1/y_1 = x_2/y_2$. Of course, since $y_1 \neq 0$ and $y_2 \neq 0$, we never multiply by 0 when we cross multiply to get $x_1y_2 = x_2y_1$, and we never divide by zero at all.

(e) Equivalence Classes: There is one equivalence class for each rational number. It consists of all the different ways to write that rational number, not necessarily reduced to lowest terms. For example,

$$\left\{\ldots, \frac{-5}{-15}, \frac{-4}{-12}, \frac{-3}{-9}, \frac{-2}{-6}, \frac{-1}{-3}, \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{13}, \ldots\right\}$$

would be one equivalence class. This is why many young people find fractions to be hard. There are infinitely many ways to write each fraction, even before you learn about negative numbers.

Note: The red entry is our usual representative when talking about this equivalence class in the rational numbers.

19. Let $\mathcal{S} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. We shall say that

$$(x_1, y_1, z_1)R(x_2, y_2, z_2) \text{ if and only if } \exists k \in \mathbb{R}^\times \text{ such that } (x_2, y_2, z_2) = (kx_1, ky_1, kz_1)$$

but for this one, don't think about the equivalences classes, because your head might explode.

(a) Reflexive: Yes. Let $k = 1$.

(b) Symmetric: Yes. Just take the reciprocal of the original k.

(c) Transitive: Yes. Suppose you use k_1 to show $(x_1, y_1, z_1)R(x_2, y_2, z_2)$ and k_2 to show $(x_2, y_2, z_2)R(x_3, y_3, z_3)$. Then $k_3 = (k_1)(k_2)$ is a good k for showing $(x_1, y_1, z_1)R(x_3, y_3, z_3)$.

(d) Equivalence Relation: Yes

Note: This is where projective geometry comes from. It was crucial in both Renaissance Art and Alchemy, and is one of the bits and pieces of Alchemy that was kept when Alchemy got rebooted into Chemistry. It is extremely important today in both modern geometry and computer graphics.

20. Let $\mathcal{S} = \mathbb{Z}$. Consider aRb if and only if a and b have the same parity. That means aRb is true if either a and b are both odd or both even; aRb is false if one is odd and the other is even.

(a) Reflexive: Yes

(b) Symmetric: Yes

(c) Transitive: Yes

(d) Equivalence Relation: Yes

(e) Equivalence Classes: There are two classes: one has all even integers; the other has all the odd integers.

21. Let $\mathcal{S} = \mathbb{Z}$. Consider aRb if and only if $a - b$ is a multiple of 3. Recall that

$$\text{multiples}(3) = \{\ldots, -15, -12, -9, -6, -3, 0, 3, 6, 9, 12, 15, \ldots\}$$

(a) Reflexive: Yes

(b) Symmetric: Yes

(c) Transitive: Yes

(d) Equivalence Relation: Yes

(e) Equivalence Classes:

$$\{\ldots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \ldots\}$$
$$\{\ldots, -11, -8, -5, -2, 1, 4, 7, 10, 13, \ldots\}$$
$$\{\ldots, -10, -7, -4, -1, 2, 5, 8, 11, 14, \ldots\}$$

Note: Clearly, aRb means $a \equiv b \bmod 3$. The red entry is our usual representative when we want to talk about one of our equivalence classes.

22. Let $\mathcal{S} = \mathbb{Z}$. Consider aRb if and only if $a - b$ is a multiple of 5. Recall that

$$\text{multiples}(5) = \{\ldots, -25, -20, -15, -10, -5, 0, 5, 10, 15, 20, 25, \ldots\}$$

(a) Reflexive: Yes

(b) Symmetric: Yes

(c) Transitive: Yes

(d) Equivalence Relation: Yes

(e) Equivalence Classes:

$$\{\ldots, -20, -15, -10, -5, 0, 5, 10, 15, 20, \ldots\}$$
$$\{\ldots, -19, -14, -9, -4, 1, 6, 11, 16, 21, \ldots\}$$
$$\{\ldots, -18, -13, -8, -3, 2, 7, 12, 17, 22, \ldots\}$$
$$\{\ldots, -17, -12, -7, -2, 3, 8, 13, 18, 23, \ldots\}$$
$$\{\ldots, -16, -11, -6, -1, 4, 9, 14, 19, 24, \ldots\}$$

Note: Clearly, aRb means $a \equiv b \bmod 5$. The red entry is our usual representative when we want to talk about one of our equivalence classes.

Note: We can consider #20 as the integers mod 2. That means aRb if and only if $a \equiv b \bmod 2$.

$$\{\ldots, -8, -6, -4, -2, 0, 2, 4, 6, 8, \ldots\}$$
$$\{\ldots, -7, -5, -3, -1, 1, 3, 5, 7, 9, \ldots\}$$

where I have highlighted in red the usual representative of each equivalence class.

23. Let $\mathcal{S}$ be the set of vertices of some graph. Consider the relation "is adjacent to." In other words $v_1 R v_2$ if and only if v_1 is adjacent to v_2.

   (a) Reflexive: No
   (b) Symmetric: Yes
   (c) Transitive: No
   (d) Equivalence Relation: No
   (e) Equivalence Classes: n/a

   Note: There is a special category of graphs, called transitive graphs, for which transitivity holds.

24. Let $\mathcal{S}$ be the set of vertices of some graph. Consider the relation "is connected to." In other words $v_1 R v_2$ if and only if $v_1 = v_2$ or if there exists a path from v_1 to v_2.

   (a) Reflexive: Yes
   (b) Symmetric: Yes
   (c) Transitive: Yes
   (d) Equivalence Relation: Yes
   (e) Equivalence Classes: The classes are the sets of vertices in each of the "connected components" of G. For example, if G is a connected graph, there is one class consisting of all the vertices.

25. Let $\mathcal{S}$ be a collection of graphs. Consider $G_1 R G_2$ if and only if G_1 is isomorphic to G_2.

   (a) Reflexive: Yes
   (b) Symmetric: Yes
   (c) Transitive: Yes
   (d) Equivalence Relation: Yes
   (e) Equivalence Classes: In each class you have a set of graphs, each isomorphic to every other in its class.

26. Let $\mathcal{S}$ be a collection of graphs. Consider $G_1 R G_2$ if and only if G_1 has the same number of vertices as G_2.

   (a) Reflexive: Yes
   (b) Symmetric: Yes
   (c) Transitive: Yes
   (d) Equivalence Relation: Yes
   (e) Equivalence Classes: A class for each positive integer. For any $z \in \mathbb{Z}^+$, you have all possible graphs with z vertices.

Note: The next five have to do with parentage. Imagine that there is a database of all birth certificates awarded in the USA in the last 200 years. The set $\mathcal{S}$ is the set of people issued birth certificates.

27. Let aRb if and only if a is the biological father of b.

(a) Reflexive: No
(b) Symmetric: No
(c) Transitive: No
(d) Equivalence Relation: No
(e) Equivalence Classes: n/a

28. Let aRb if and only if a and b have both biological parents in common.
    (a) Reflexive: Yes
    (b) Symmetric: Yes
    (c) Transitive: Yes
    (d) Equivalence Relation: Yes
    (e) Equivalence Classes: Each class is a set of full siblings. In other words, a set of persons who are all brothers/sisters.
29. Let aRb if and only if a and b have one biological parent in common.
    (a) Reflexive: Yes
    (b) Symmetric: Yes
    (c) Transitive: No
    (d) Equivalence Relation: No
    (e) Equivalence Classes: n/a
30. Let aRb if and only if a is a biological ancestor of b.
    (a) Reflexive: No
    (b) Symmetric: No
    (c) Transitive: Yes
    (d) Equivalence Relation: No
    (e) Equivalence Classes: n/a
31. Let aRb if and only if a and b have the same biological mother.
    (a) Reflexive: Yes
    (b) Symmetric: Yes
    (c) Transitive: Yes
    (d) Equivalence Relation: Yes
    (e) Equivalence Classes: A class for each mother, consisting of all her children.
32. Let $\mathcal{S}$ be any set. Let aR_1b be an equivalence relation on $\mathcal{S}$. Define aR_2b to be the negation of aR_1b. In other words, aR_2b is true if and only if aR_1b is false.
    (a) Reflexive: No
    (b) Symmetric: Yes, but it is hard to see it.
    Proof: Assume R_2 is not symmetric. Then there must exist some a and some b such that aR_2b is true, and bR_2a is false. However, that would mean that aR_1b is false, and bR_1a is true. This would imply R_1 is not symmetric. Yet, we said R_1 is an equivalence relation, so it is symmetric. This is a contradiction. Therefore, our assumption that R_2 is not symmetric must be false. In conclusion, R_2 is symmetric. □

(c) Transitive: No
(d) Equivalence Relation: No
(e) Equivalence Classes: n/a

33. Let $\mathcal{S}$ be any set. Let aR_1b and aR_2b be two equivalence relations on $\mathcal{S}$. Define aR_3b to be the union of aR_1b and aR_2b. In other words

$$(aR_3b) \Leftrightarrow [(aR_1b) \vee (aR_2b)]$$

(a) Reflexive: Yes
(b) Symmetric: Yes
(c) Transitive: No
(d) Equivalence Relation: No (additional notes follow below)
(e) Equivalence Classes: n/a

34. Let $\mathcal{S}$ be any set. Let aR_1b and aR_2b be two equivalence relations on $\mathcal{S}$. Define aR_3b to be the intersection of aR_1b and aR_2b. In other words

$$(aR_3b) \Leftrightarrow [(aR_1b) \wedge (aR_2b)]$$

(a) Reflexive: Yes
(b) Symmetric: Yes
(c) Transitive: Yes
(d) Equivalence Relation: Yes (additional notes follow below)
(e) Equivalence Classes: Each class C consists of a set of objects such that aR_1b is true and aR_2b is also true, for all $a \in C$ and $b \in C$.

On the Transitivity of the Intersection and the Union

Claim: If R_1 and R_2 are equivalence relations on $\mathcal{S}$, then their intersection, R_3, is transitive.

Proof: It suffices to prove that if R_1 and R_2 are equivalence relations on $\mathcal{S}$, R_3 is the intersection of R_1 and R_2, and both aR_3b and bR_3c are true (for some $a, b, c \in \mathcal{S}$), then aR_3c is true.

Suppose R_1 and R_2 are equivalence relations on $\mathcal{S}$, that R_3 is the intersection of R_1 and R_2 and that aR_3b and bR_3c are both true (for some $a, b, c \in \mathcal{S}$).

Since aR_3b is true, then aR_1b and aR_2b are both true.
Since bR_3c is true, then bR_1c and bR_2c are both true.
Since R_1 and R_2 are equivalence relations, they are transitive.
Since R_1 is transitive, both aR_1b and bR_1c are true, then aR_1c is true.
Since R_2 is transitive, both aR_2b and bR_2c are true, then aR_2c is true.
Since both aR_1c and aR_2c are true, and R_3 is the intersection of R_1 and R_2, then aR_3c is true.

Therefore, if R_1 and R_2 are equivalence relations on $\mathcal{S}$, R_3 is the intersection of R_1 and R_2, and aR_3b and bR_3c are both true (for some $a, b, c \in \mathcal{S}$), then aR_3c is true.

Equivalently, if R_1 and R_2 are equivalence relations on $\mathcal{S}$, then their intersection, R_3, is transitive. □

(Discussion continues on the next page. . .)

What About the Union? You might recall that I claimed the intersection is transitive, but the union is not transitive. Look at the above proof. The phrase "aR_1b and aR_2b are both true" would become "aR_1b or aR_2b is true." Similarly, the phrase "bR_1c and bR_2c are both true" would become "bR_1c or bR_2c is true."

It might be the case that aR_1b is true but aR_2b is false. It might also be the case that bR_2c is true but bR_1c is false. In this case, there is no way for us to use the transitivity of R_1 and R_2 to force aR_3c to be true. We cannot use the transitivity of R_1, because bR_1c is false; we cannot use the transitivity of R_2 because aR_2b is false. In conclusion, we cannot modify the above proof to work for the union.

However, this does not address the matter completely. What if there is an unrelated way to prove that the union of two equivalence relations is transitive? The only way that can I convince you that no such proof could be written would be to produce a counter-example.

- Let $\mathcal{S}$ be the integers.
- Let aR_1b be true if and only if $a - b$ is a multiple of three. (See #21.)
- Let aR_2b be true if and only if $a - b$ is a multiple of five. (See #22.)
- Let aR_3b be true if and only if either aR_1b or aR_2b.
- Let $a = 15$, $b = 10$, and $c = 7$.
- Since $15 - 10 = 5$ is not a multiple of 3, we know that aR_1b is false.
- Since $15 - 10 = 5$ is a multiple of 5, we know that aR_2b is true.
- Since aR_2b is true, then aR_3b is true.
- Since $10 - 7 = 3$ is a multiple of 3, we know that bR_1c is true.
- Since $10 - 7 = 3$ is not a multiple of 5, we know that bR_2c is false.
- Since bR_1c is true, then bR_3c is true.
- Since $15 - 7 = 8$ is not a multiple of 3, we know that aR_1c is false.
- Since $15 - 7 = 8$ is not a multiple of 5, we know that aR_2c is false.
- Since both aR_1c is false and aR_2c is false, then aR_3c is false.
- We have shown aR_3b is true, bR_3c is true, and aR_3c is false.
- Therefore, R_3 is not transitive.
- Furthermore, R_3 is not an equivalence relation.

Sage Code for the Image in Problem #8

If you are curious, then the code for making the graphic (in Sage) for Problem # 8 is given below:

```
P=Graphics()

for k in range(-2, 5):
    temp_x=2*pi*k + pi/6
    P = P + point( (temp_x,1/2), size=100, color="red" )
    temp_x=2*pi*k + 5*pi/6
    P = P + point( (temp_x,1/2), size=100, color="red" )

plot(sin(x), x, -13, 28, ymin=-2, ymax=2) + plot(P) + plot(
    1/2, x, -13, 28, color="green", gridlines="minor")
```