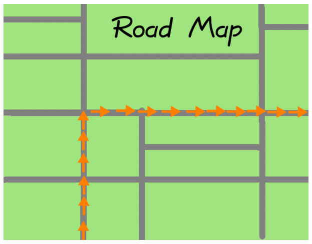
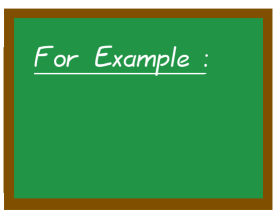
Module 3.4: Expected Value and Insurance



The concept of expected value is rather simple but extremely useful. Typically, it is omitted from discrete mathematics courses because the topic is not difficult. In fact, I omitted the topic myself for the first times that I taught the course.

However, I came to realize that it should be included, because it usefully describes several important phenomena, including insurance and stock options. Not everything in life has to be difficult.



3-4-1

Let's suppose that you work for a company that manufactures remote-control drones. One model of drone has four propellers that keep the drone aloft. Unfortunately, something has gone horribly wrong in the manufacturing process, and some products have faulty propellers. Replacement propellers can be ordered, but we need to know how many propellors we should expect to need for the next 1,716,000 drones. The first 10,000 drones are inspected, and here is the data:

98.02%	all propellers okay
1.70%	one faulty propeller
0.23%	two faulty propellers
0.03%	three faulty propellers
0.02%	all propellers broken

We are going to make a table of four rows. The first row is the name of the event; the second row is the probability; the third row is the number of broken propellers; the fourth row is the probability multiplied by the number of broken propellers.

Event	All Okay	One Faulty	Two Faulty	Three Faulty	All Broken
Probability	0.9802	0.0170	0.0023	0.0003	0.0002
Value	0	1	2	3	4
Product	0	0.0170	0.0046	0.0009	0.0008

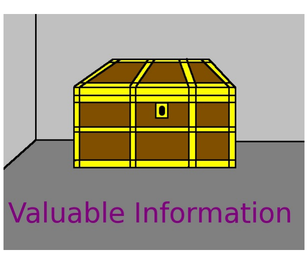
By adding up the bottom row, we learn that the expected value is

$$0 + 0.0170 + 0.0046 + 0.0009 + 0.0008 = 0.0233$$

broken propellors per drone, which means we can expect that we will need

$$(0.0233)(1,716,000) = 39,982.8$$

replaced propellers. This information will be valuable to whomever places the order.



The procedure done in the previous example works for almost all examples relating to expected value. I like to call this "the method of four rows."

1. The first row has each of the outcomes (simple events) in the sample space.
2. The second row has the probabilities of those outcomes.
3. The third row has a numerical value associated with each outcome (simple event). This is often a monetary value, but as we saw in the previous example, it does not have to be. Since monetary values are the most common, the entries of this row are often called "the payoffs" of the outcomes.
4. The fourth row has the probability times the payoff for each outcome (simple event).
5. The expected value is the sum of the fourth row.

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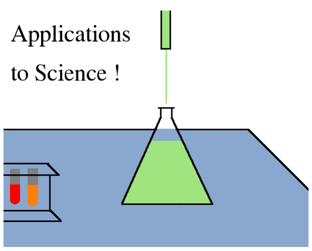
Many of my readers have had a course in physics or multivariate calculus, and have learned about vectors. However, many of my readers have not yet had that course. If you don't know about vectors, then skip to the next box at this time.

If you happen to know what computing a dot-product of two vectors is about, then you have already noticed that the expected value is actually a dot-product of two vectors. There is a vector of probabilities, and a vector of payoff values.

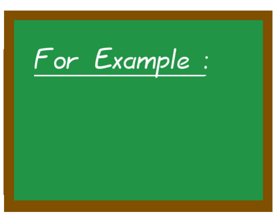
We will not explore the connections with vectors in this textbook any further. However, in quantum mechanics, there are further connections between vectors and probability.



The next few problems are meant to introduce some of the difficulties that arise when considering questions in the social sciences. The question of how many members of the population of a country have been married 3 times, 2 times, 1 time, or 0 times seems remarkably straightforward at first glance. Yet, even this simple question requires some degree of attention to detail.

The data on marriage which I will be quoting comes from a 17-page report published in February 2005 by the United States Census Bureau, based on the 2000–2001 census. Of course, the 2020–2021 census isn't finished yet—as I am writing this in October of 2020, and I do not believe that this specific question has been analyzed during the 2010–2011 census, but I could be wrong.

You can see the entire 17-page report at this link:
`https://www.census.gov/prod/2005pubs/p70-97.pdf`



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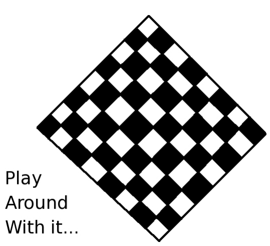
Looking at the males in the previously mentioned report, there were 53.4% of men who had been married exactly once; there were 12.5% of men who had been married twice; there were 3.2% of men who had been married three or more times; there were 30.9% of men who had never been married. What is the expected value of the number of times a random male has been married?

Unfortunately, the "three or more times" is a stumbling block to our calculation. There could be someone who had married four times, or even five or six times. However, as the report itself stated, persons who had been married four or more times are much less than 1% of the population, so that can be neglected.

Outcome	Never Married	Married Once	Married Twice	Married 3 or more times
Probability	0.309	0.534	0.125	0.032
Marriages	0	1	2	3
Product	0	0.534	0.250	0.096

The expected value of the number of marriages of a random male survey respondent is

$$0 + 0.534 + 0.250 + 0.096 = 0.880$$



3-4-3

For practice, I'd like you to solve a problem similar to the previous example, but with the data for females. There were 58.7% of women who had been married once; 13.6% of women who had been married twice; 3.1% of women who had been married three or more times; 24.6% of women who had never been married.

What is the expected value of the number of marriages for a random female survey respondent? The answer will be given on Page 320.

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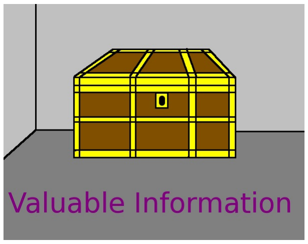


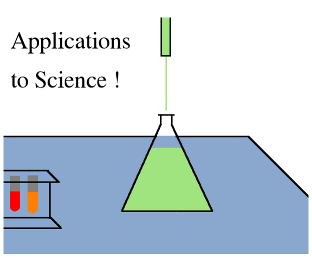
I'd like to clarify some vocabulary.

- We begin with a set of outcomes that are mutually exclusive and collectively exhaustive. This is a *sample space*, and is the first row. This was also defined earlier, on Page 233 in Module 3.1: "A Formal Introduction to Probability Theory."
- Then we assign, measure, or compute probabilities for those outcomes. That is the second row. The first row and the second row—a sample space with probabilities assigned—when considered together are called a *probability distribution*. This was also defined earlier, on Page 242 in Module 3.1: "A Formal Introduction to Probability Theory."
- Next, we associate a number, which is frequently a dollar value but it can be any number appropriate to the situation at hand, to each outcome. That is the third row. The first three rows taken together—a sample space where each outcome has both a probability and a numerical value—is called a *random variable*.

Mixing up vocabulary terms during a job interview is a great way to sound like a fool.

- When digital cameras were new, people with only intermediate knowledge of computers would confuse "How many megapixels does it have?" with "How many megabytes can it store?"
- Even today, some people with only intermediate knowledge of computers will interchange gigahertz with gigabytes.
- Another great example was an email exchange between one of my undergraduate advisees and a company he wanted to get a job with. He was making good progress, and then wrote "Microsoft XL" instead of "Microsoft Excel." He did not hear back.
- Similarly, a company might be excited to hire a new college graduate, because you are supposed to be "the data-science generation." Imagine how devastating it would be if you were, while talking quickly, to interchange the terms "random variable" with "expected value," or the terms " probability distribution" with "sample space."

Such mistakes can torpedo your candidacy for a job.



I should emphasize that our calculations about marriage were only possible because persons who had been married four or more times are rare. This permitted us to represent the "3 or more" column as simply the number "3."

In contrast, there was a survey that I was looking over, about the prevalence of various forms of sexual activity among high-school students, which had been conducted to combat teenage pregnancy and sexually transmitted diseases. It was expensive research to perform, involving many schools, and thousands of students.

The highest category was "20 or more times per year," and for some activities, a significant number of students had chosen that. Perhaps one student does something three times per week—that's 156 times per year. Another student does that thing about twice a month—that's 24 times a year. Yet, both would have selected "20 or more times per year."

This ambiguity completely destroys our ability to compute an expected value. All is not lost, because we can still compute the probability of a random student doing that activity 20 or more times per year. However, the expected value was a number that would have been necessary in certain risk calculations. Those risk calculations could not be performed, because the survey had this design flaw.

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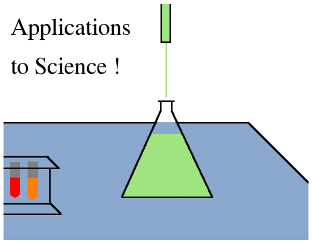
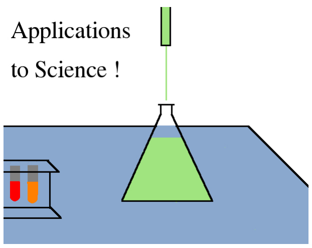


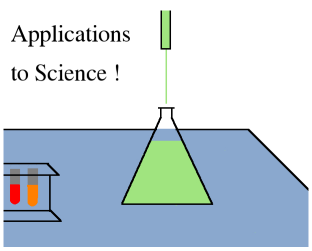
Another issue that should be discussed is the distinction between "a random citizen," versus "a random survey respondent." I'm going to offer two examples.

Even as recently as 2009, a significant number of low-income households had no internet access. If you run a survey through the internet, then low-income households will be underrepresented. Probably, some low-income households do have internet access, of course. Yet, nearly all middle-income and high-income households have internet access. The lack of universal internet access among low-income households will make their responses be a smaller proportion of the set of survey responses compared to the proportion of low-income households among the set of all households. This will definitely skew the data. This phenomenon is often called "sample bias."



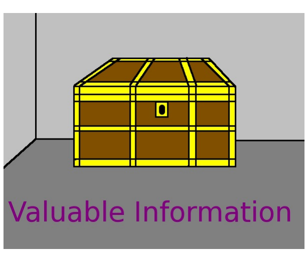
Surveys of sexual activity among high-school or college students are particularly prone to something called response bias. Imagine someone who has sworn a virginity pledge, and who eschews all (or almost all) sexual activities for religious reasons. Among some Orthodox Jewish communities, such as Chabad-Lubavitch, young people who are dating are forbidden to hug, to shake hands, or even to give the briefest kiss on the cheek, until marriage.

It is likely that some students of that background would be disgusted by a survey of sexual activity, and would simply decline to fill it out. This will remove a large number of "0 times per year" responses from the expected-value computation. That would gravely skew the results, essentially making the results useless.



At the opposite end of the spectrum, it is also easy to imagine, especially in a high-school setting, some younger students who invent sexual experiences that they haven't actually had. They might respond thinking about the type of life that they wished they were living, instead of the type of life that they are actually living. These exaggerations (or outright fictions) can ruin the calculation, if a significant number of students respond in that manner.

Among mid-career professionals, data about salaries and bonuses has been shown to be similarly exaggerated. Bonuses, in particular, have been shown to be vulnerable. If a mid-career professional is likely to get a bonus, especially if they have often received it in the past, then they will report that bonus as if it had already been awarded. This makes their total compensation (salary plus bonuses) appear larger than it actually is.



Perhaps you might be wondering what the real "take-away message" of the previous four boxes happens to be. Permit me to summarize. Research that is performed in the social sciences based entirely upon survey responses is vulnerable to the following difficulties:

- Sample bias.
- Response bias.
- Exaggerations or fictional responses.

Any particular survey should be analyzed with an eye toward the methodology. There are strategies to combat these difficulties. However, if none of those strategies were used, then no major policy decisions should be made on the basis of that survey.

Of course, there are other avenues for conducting research in the social sciences, such as analyzing tax returns, standardized tests in schools, or going to clinics to ask about how many patients had been diagnosed with a particular sexually-transmitted disease.

It is definitely false to say that all research in the social sciences is unreliable. Instead, it is research based entirely (or mostly) on survey responses of human beings that is unreliable.

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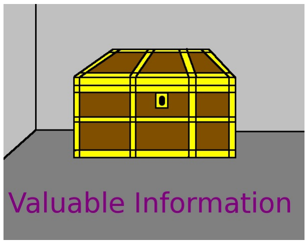
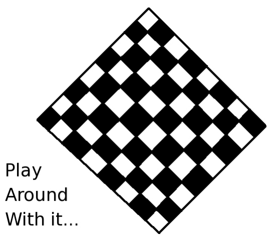


One use of expected value is to know what you can expect of a random variable. A second use deals with insurance, and we'll explore that in detail, later in this module. A third use has to deal with totaling many samples of a random variable.

Let X be a random variable with expected value $E[X]$. Suppose you sample from a random variable X, a total of n times, and add all the samples up. Call the total T. Then, in the limit as n goes to infinity, T goes to $nE[X]$.

This means that for sufficiently large n, if the only thing you care about is the total, then you can pretend as though X were approximately a constant, roughly equal to $E[X]$.

Note, we quietly took advantage of this fact on Page 301, in the first example of this module, when we analyzed the remote-control drones and their broken propellors.

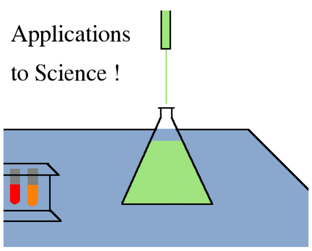


3-4-4

Imagine a publishing house that is sending English-language books to Greece. In Greece, the English language is taught starting in 2nd grade, so many Greeks are excellent with English. However, the distance is pretty far. New York City to Athens is 4921 miles or 7920 kilometers, approximately an 11-hour flight. Shipping across such enormous distances makes weight a critical variable.

Suppose you're hired to compute the expected value of the weight of a typical book that has to make the trip. You ask for six-months worth of data, and learn that 117 books weigh 2.8 pounds; 328 books weigh 2.1 pounds; 847 books weigh 1.5 pounds; and 1134 books weigh 0.6 pounds.

Make a table suitable for the use of the method of four rows, and then compute the expected value. The answer will be given on Page 320 of this module.



Of course, in Greece, they use kilograms instead of pounds. Do we need to recompute the expected value, using the method of four rows, for the weight of the books in kilograms?

Intuitively, we know that the answer is no. We can look up the conversion for pounds to kilograms, and multiply by the appropriate number. There is no reason to repeat the expected-value computation from scratch.

It turns out that one pound (at sea level) is 0.45359237 kilograms. Therefore, we just have to multiply.

$$(1.22312\cdots \text{ lb/book})\,(0.453592\cdots \text{ kg/lb}) = 0.554797\cdots \text{ kg/book}$$

The general way of stating this result is $E[kX] = kE[X]$, for any constant k. However, k must be a constant. We will not prove this result now, because it is merely a special case of a theorem that we will prove a few pages from now.

After all these examples, you've seen that the expected value is a single number. Shortly, we will see that it can also be a compact formula containing a variable for an unknown probability or unknown payoff. Even when there is a variable in there, the expected value becomes a single number once that variable becomes known.

I have seen a moderately large number of students over the years report the entire fourth row as the expected value. That's wrong, because the expected value is not a sequence of numbers—instead, it is a single number. This sort of mistake is damaging in a research project or even a highly technical job interview, because it reveals that the student has no idea what expected value is even about.

While most technical job interviews in computing draw their questions from the courses typically called *Data Structures*, or *Algorithms*, a fairly large number of questions come from *Discrete Mathematics* as well.

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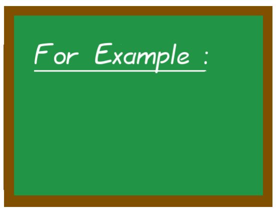
We're now going to compute $E[X^2]$ and $E[X]^2$ for two examples. It probably isn't obvious why anyone would want to do that, but we'll discuss the reasons later.

I think many of my readers might assume that

$$E[X]^2 = E[X^2]$$

but this is not the case in general.



3-4-5

Earlier, on Page 301 of this module, we computed the expected value of the number of broken propellors on a 4-propellor drone. If that is $E[X]$, then suppose that someone wanted to know $E[X]^2$ and $E[X^2]$.

For $E[X]^2$, we can just square the answer we found earlier, and we obtain

$$E[X]^2 = (0.0233)^2 = 0.00054289$$

but we need to make a new table for $E[X^2]$.

Permit me to share that I found it amusing that out of 36 entries in the table from that page, only 6 entries required making any changes at all. The table is in the next box, and those entries are marked with a $\star$.

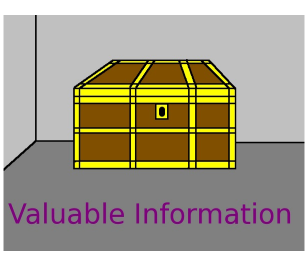
Continuing with the next box, here is the four-row table for $E[X^2]$.

Event	All Okay	One Faulty	Two Faulty	Three Faulty	All Broken
Probability	0.9804	0.0170	0.0023	0.0003	0.0002
X^2	0	1	$\star$4	$\star$9	$\star$16
Product	0	0.0170	$\star$0.0092	$\star$0.0027	$\star$0.0032

We can find the expected value by summing up the fourth row

$$E[X^2] = 0 + 0.0170 + 0.0092 + 0.0027 + 0.0032 = 0.0321$$

and we see that $E[X^2] \neq E[X]^2$ in this case, since $E[X]^2 = 0.00054289$.



Since we've found one random variable where $E[X^2] \neq E[X]^2$, clearly it is obvious that the equation $E[X^2] = E[X]^2$ is not true in general.

We can now confidently conclude that

$$E[X^2] = E[X]^2 \text{ is false in general}$$

which could be also written as

$$E[(X)(X)] = (E[X])(E[X]) \text{ is false in general}$$

However, we have postponed the question of why would anyone ever want to know $E[X^2]$?! We'll discuss that in the next box.

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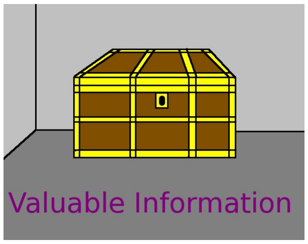


It turns out that $E[X^2] \geq E[X]^2$ holds true for all random variables without exception, a theorem we could prove, but which we will not prove right now.

Moreover, the difference

$$E[X^2] - E[X]^2 = Var[X]$$

is so important, that it gets a name. That's called the *variance* of the random variable X.

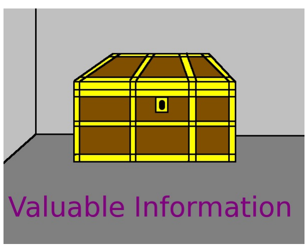
The square root of the variance is called the *standard deviation*. The variance and the standard deviation are (by leaps and bounds) the most common measures of the dispersion of a random variable. The dispersion means "how spread out" it is.

Of course, if you trust me that $E[X^2] \geq E[X]^2$ holds true for all random variables without exception, then you know that

$$E[X^2] - E[X]^2 \geq 0$$

This means that the variance of a random variable is never negative, therefore the variance is always either zero or positive. Similarly, since the standard deviation is the square root of the variance, it is always either zero or positive.

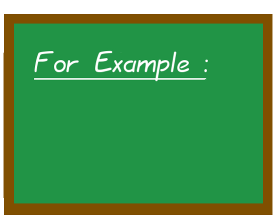
The concepts of variance and standard deviation are straightforward, but they deserve a module of their own. Therefore, we won't discuss them further here.



Meanwhile, you might be curious if there are some random variables where $E[X^2] = E[X]^2$.

As it turns out, $E[X^2] = E[X]^2$ if and only if X is a constant. We say that a random variable X is a constant if and only if it has exactly one outcome with a non-zero probability. Since the sum of the probabilities must equal 1, that sole outcome with non-zero probability must have a probability equal to 1.

The proof of this fact is not too hard, but we will not explore it at this time.



3-4-6

Suppose that the army of a newly independent country needs size data for making uniforms. They get the height of several battalions of soldiers, perhaps a few thousands soldiers, measured. Let n be the number of soldiers, and let their heights be h_1, h_2, h_3, $\ldots$, h_n. What is the expected value of their height?

Specifically, the process we must imagine is randomly selecting one of the n soldiers, and asking them their height. The probability of selecting any particular soldier is $1/n$.

Event	Soldier 1	Soldier 2	Soldier 3	$\cdots$	Soldier n
Probability	$1/n$	$1/n$	$1/n$	$\cdots$	$1/n$
Value	h_1	h_2	h_3	$\cdots$	h_n
Product	h_1/n	h_2/n	h_3/n	$\cdots$	h_n/n

Therefore, the expected value equals the following sum:

$$\frac{h_1}{n} + \frac{h_2}{n} + \frac{h_3}{n} + \cdots + \frac{h_n}{n} = \frac{h_1 + h_2 + h_3 + \cdots + h_n}{n}$$

Look, that's something extremely familiar to us all! It is just the average height of all the soldiers! How do you compute an average? You add up all the numbers in your data set, and divide by how many numbers you have in your data set. The expected value is exactly that average.

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Suppose that the height measurement of the previous box was done during a physical exam of the soldiers, when they all would be wearing little and be barefoot. The designer of the uniforms might have to explain that the height should have been taken when the soldiers were wearing their boots, because the uniform will be worn with boots in practice.

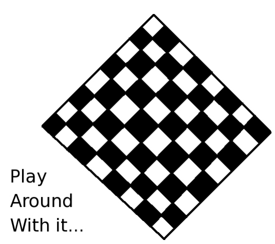
For ease of computation, let us suppose that the boot heel is 3 cm, so each soldier becomes precisely 3 cm taller when wearing the boots compared to not wearing the boots. How does this change the expected value?

Clearly, h_1 should be replaced with $(h_1 + 3)$, and similarly for h_2, h_3, $\ldots$, h_n.

$$\begin{aligned}
E[H+3] &= \frac{h_1+3}{n} + \frac{h_2+3}{n} + \cdots + \frac{h_n+3}{n} \\
&= \frac{(h_1+3)+(h_2+3)+\cdots+(h_n+3)}{n} \\
&= \frac{h_1+h_2+\cdots+h_n+\overbrace{3+3+\cdots+3}^{n \text{ copies of } 3}}{n} \\
&= \frac{h_1+h_2+\cdots+h_n+3n}{n} \\
&= \frac{h_1+h_2+\cdots+h_n}{n} + \frac{3n}{n} \\
&= \frac{h_1+h_2+\cdots+h_n}{n} + 3 \\
&= E(H)+3
\end{aligned}$$

As you can see, there is no reason to measure everyone's height again (with boots) if all we care about is the expected value. The expected value simply increases by 3. Shortly, we will see that this always happens.

To be precise, if c is a constant and X is a random variable, then $E[X+c] = E[X]+c$. We will not prove this, because it is an easy consequence of a more general theorem that we will prove momentarily.



3-4-7

Let's consider the soldiers again, but instead of having the random variable H be their heights (without boots), let the random variable T be how much taller or shorter they are than the mean (without boots). Suppose the mean is 178 cm, and we'll keep all the height measurements without boots.

- If Ned is 180 cm, which is 2 cm taller the mean, we record his T-entry as $+2$.
- If Fred is 185 cm, which is 7 cm taller than the mean, we record his T-entry as $+7$.
- If Ted is 175 cm, which is 3 cm shorter than the mean, we record his T-entry as -3.

Do we know what the expected value of T will be? Or do we have to go and measure all the soldiers again?

This question is a bit harder than the last few, but it is worth it to pause, think about it a bit, and only look at the answer after having thought about it. The answer is given momentarily.

It is important not to read the answer, given below, without having first made an honest attempt to think about the question of the previous box.

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Here is the answer to the previous checkerboard problem, about the random variable T, how much taller (or shorter) a soldier is than the mean of 178 cm.

One key detail is to remember that H was the random variable of the height of a soldier. We know that $E[H] = 178$ cm. Another key detail is the fact that for any constant c, and any random variable X, we know

$$E[X + c] = E[X] + c$$

but c must be a constant.

We must realize that $T = H - 178$. So we can apply the $E[X + c] = E[X] + c$ formula with $X = H$ and $c = -178$. We obtain

$$E[H - 178] = E[H] + (-178) = 178 - 178 = 0$$

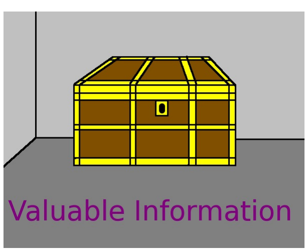
therefore, $E[T] = E[H - 178] = 0$.

The phenomenon we saw in the previous box always happens. To be precise, if you take the values of a random variable, and subtract the expected value from each outcome's value to construct a new random variable, then your new random variable will always have expected value equal to zero. This process is not very common but not very rare, and it is called *centralizing a random variable*.

Formally, if X is a random variable with an expected value of μ, then $E[X - \mu] = 0$. This is often written as

$$E[X - E[X]] = 0$$

We will not prove this specific result, because it is just a special case of a general theorem, which we are now ready to prove.



If there is a random variable R, and a function $f(x) = mx + b$, where m and b are constants, then

$$E[f(R)] = f(E[R])$$

However, the requirement that m and b are constants is absolutely essential.

Claim: If there is a random variable R, and a function $f(x) = mx + b$, where m and b are constants, then $E[f(R)] = f(E[R])$.

Proof: Let R be a random variable with n outcomes. Suppose that the probabilities of the n outcomes for the random variable R equal p_1, p_2, p_3, ..., p_n. Further suppose that the payoff values of the n outcomes for the random variable R equal v_1, v_2, v_3, ..., v_n.

This means that the payoff values of the n outcomes for the random variable $f(R)$ equal $mv_1 + b$, $mv_2 + b$, $mv_3 + b$, ..., $mv_n + b$.

We will continue in the next box.

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Continuing with the previous box, the fourth row for $f(R)$ equals $p_1(mv_1+b)$, $p_2(mv_2+b)$, $p_3(mv_3+b)$, $\ldots$, $p_n(mv_n+b)$, and the expected value $E[f(R)]$ is the sum of the fourth row.

$$\begin{aligned}
E[f(R)] &= p_1(mv_1+b)+p_2(mv_2+b)+p_3(mv_3+b)+\cdots+p_n(mv_n+b)\\
&= p_1mv_1+p_1b+p_2mv_2+p_2b+p_3mv_3+p_3b+\cdots+p_nmv_n+p_nb\\
&= p_1mv_1+p_2mv_2+p_3mv_3+\cdots+p_nmv_n+p_1b+p_2b+p_3b+\cdots+p_nb\\
&= m\left(p_1v_1+p_2v_2+p_3v_3+\cdots+p_nv_n\right)+\left(p_1+p_2+p_3+\cdots+p_n\right)b\\
&= m\left(p_1v_1+p_2v_2+p_3v_3+\cdots+p_nv_n\right)+(1)\,b\\
&= m\left(E[R]\right)+b\\
&= f(E[R])
\end{aligned}$$

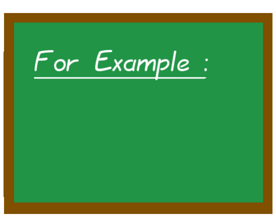
In conclusion, $E[f(R)] = f(E[R])$, for any random variable R, and for any function $f(x) = mx+b$, provided that m and b are constants. ■

Having proven that theorem, we can easily show why, for any random variable R, we know that

$$E[R-E[R]] = 0$$

Let $\mu = E[R]$ and let $f(x) = x-\mu$. Now apply the theorem that we just proved. We obtain

$$E[R-E[R]] = E[R-\mu] = E[f(R)] = f(E[R]) = E[R]-\mu = \mu-\mu = 0$$



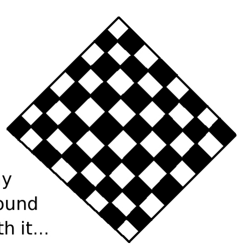
3-4-8

On Page 236 of Module 3.1: "A Formal Introduction to Probability Theory," we introduced the concept of a Bernoulli random variable. It is traditional to describe it as an unfair coin. There is a probability p for some event $\mathcal{B}$, and if you make n trials, you expect $\mathcal{B}$ to happen np times.

We can show why np is the expected value for the number of successes, by simply denoting each success ($\mathcal{B}$) as 1, and each non-success ($\mathcal{B}^c$) as 0.

Outcome	$\mathcal{B}$ happened	$\mathcal{B}$ didn't happen
Probability	p	$1-p$
Value	1	0
Product	p	0

As you can see, the expected value is $p+0=p$. Therefore, if the probability of $\mathcal{B}$ is p, and you try n times, you expect $\mathcal{B}$ will happen np times.



Play Around With it...

3-4-9

Continuing with the previous box, if X is a Bernoulli random variable that is 1 with probability p and 0 with probability $1-p$, then compute the following:

- What is $E[X^2]$?
- What is $E[X]^2$?

The answers will be given on Page 321 of this module. There's also a surprising connection between this question and the Bernoulli-DeMoivre-Laplace Inequalities, which will be described for you in the box containing the answers to this question.

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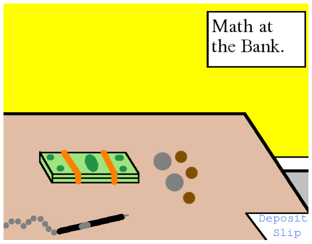
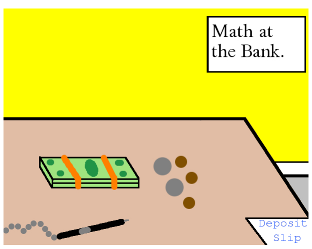


Now we turn to the classic application of expected value: insurance. The basic idea of insurance is simple. Consumers buy insurance to protect themselves from financial losses due to events called *hazards*.

- For homeowner's insurance, the house could be destroyed by hazards such as by fire, by earthquake, by landslide, by a truck driving through it if near a highway, or by flood. It is worthwhile to know that flood insurance is sold separately from the standard homeowner's insurance that covers the other hazards.
- For car insurance, there are several hazards that can cause you to lose your car. The most obvious include collisions, vandalism, and autotheft, but there are others, such as fire. It is noteworthy that in the USA, the owner of a car can choose to cover only some hazards or perhaps all hazards. There can also be some coverage limits, which become a bit complicated. For simplicity of the explanation, I will describe a comprehensive car-insurance policy, which covers all hazards, when we discuss car insurance below.
- For trip insurance, there are several hazards that could cause your trip to be ruined. You could get sick, a dependent not traveling with you could get sick, or you could get into a car accident on the way to the airport. Perhaps your flight could get cancelled, causing you to miss a long and expensive cruise.

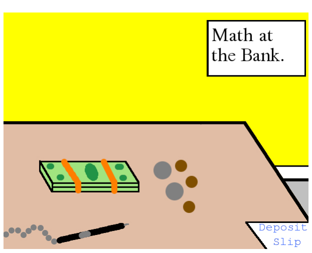


Let's focus just for a moment on car insurance, since it is the form of insurance that most of my students have. If I were to get into a car accident, then my insurance company would pay for the damages to my car, as well as the damages to the other car, and if needed, any personal injury damages.

However, from this total payout (which could be a very large check), they would deduct \$ 1000. That's because the policy that I have has a *deductible* of \$ 1000 specified in writing. Other common sizes of deductible in the USA include \$ 500, \$ 250, \$ 100, or even \$ 2000.

There is also a *premium*, which is the amount that I pay the insurance company, each month, in return for this protection. The premium can also be paid annually or quarterly. The insurance company collects the premium from all of its customers, over and over again, and that is how it has money to pay the bills of any customers who get into an accident.

It is clear that the sum of the premiums should exceed the sum of the checks paid out to customers. The excess will be used to pay the employees of the insurance company, other expenses of the company, and profit for the insurance company's shareholders.



Now that we understand car insurance, there are a few other types of insurance you should know about.

- Homeowner's insurance works essentially the same way as comprehensive car insurance.
- Jewelry and artwork are often protected by theft insurance, which also works like car and homeowner's insurance.
- Speaking of theft, if you own several pieces of computing and gaming technology, then you might want to consider renter's insurance to protect those items from theft, including other hazards like fire or vandalism. Recent college graduates are often surprised to learn how affordable this coverage can be.
- Trip insurance differs only slightly from these in that the premium is paid only once, when the airline tickets or cruise tickets are booked. In the previous cases, the premium would be paid monthly, quarterly, or annually, in the USA.

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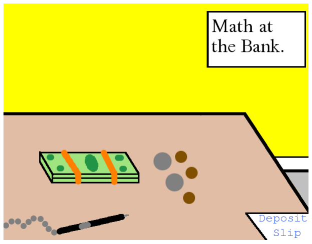


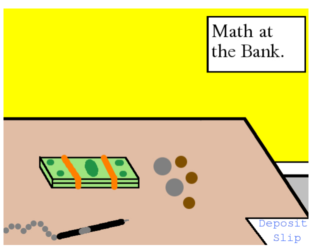
The fundamental strategy of insurance, from the well-informed consumer's perspective, has been taught the same way for the last 300–400 years. First, I ask myself: is it possible for me to absorb the loss without catastrophic inconvenience?

If the answer is no, then I should get the insurance. Of course, I might shop around a bit among competing insurance companies to find the best deal, but I should not go without insurance.

If the answer is yes, then I should still compute the expected value. (I will show you how to do that momentarily.)

- If the expected value is positive, then I should get the insurance.
- However, if the expected value is negative, then I should not get the insurance.
- Lastly, when the expected value is zero, the term is that I am "ambivalent" about the insurance. Ambivalent is a professional term that means I don't care, but it is more polite.

It is worth mentioning that some insurance is simply required by law. In those cases, of course the insurance should be obtained, but I might be wise to shop around a bit among competing insurance companies, to get a good price.

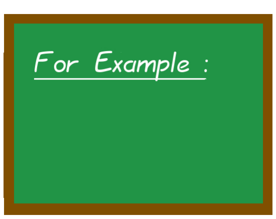


In the previous box, we left catastrophic inconvenience undefined. This is not a mathematical term with a precise definition. Instead it is a concept that is best described by an example.

For many typical workers, if they get into a car accident and their car is demolished, then they will be unable to get to and from their place of work. Typically, this could result in losing the job entirely. The worker might have enough savings to pay rent for a month or three while looking for a new job. If not, or if it takes too long to find a new job, then they might become homeless.

Alternatively, perhaps they might have enough savings to buy a cheap used car on short notice. Yet, this depletion of their savings might make them unable to pay the next month's rent, and they might become homeless for that reason. That's the scale of a catastrophic inconvenience.

At the opposite end of the scale, would be a mid-career professional getting trip insurance on airline tickets. If it is a few hundred dollars, the loss might be irritating, but it will not cause a major catastrophe such as becoming homeless or losing a job.



3-4-10

On December 3rd, 2018, I was booking a flight to Baltimore. The airfare came to \$ 617.60. Delta Airlines offered me trip insurance with a premium of \$ 41.69. If I were to be unable to fly due to illness, or other hazards such as the illness of a loved one, then the trip insurance would reimburse me for the cost of the airfare. A primitive way to describe this situation is to assume that the probability of this hazard is some fixed number, such as 5%.

Outcome	Too Sick to Travel	Okay to Travel
Probability	0.05	0.95
Payoff	$617.60 - 41.69 = 575.91$	-41.69
Product	$(575.91)(0.05) = 28.7955$	$(-41.69)(0.95) = -39.6055$

The expected value of this trip insurance is

$$28.7955 - 39.6055 = -10.81$$

and since it is negative, I should probably decline. An insurance policy is considered attractive if and only if the expected value is positive.

However, this decision-making is deeply flawed, because the number 5% was simply made up. Instead, I'll show you how to do the calculation as a professional would, in the next box.

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Instead of arbitrarily choosing 5% for the probability that I will be too sick to travel, we will let that probability remain unknown, and denote it with the variable p. The table is only slightly more complicated with this change.

Outcome	Too Sick to Travel	Okay to Travel
Probability	p	$1-p$
Payoff	$617.60 - 41.69 = 575.91$	-41.69
Product	$(575.91)(p)$	$(-41.69)(1-p)$

The expected value of this trip insurance is

$$(575.91)(p) + (-41.69)(1-p) = 575.91p - 41.69 + 41.69p = 617.6p - 41.69$$

For Example :

3-4-11

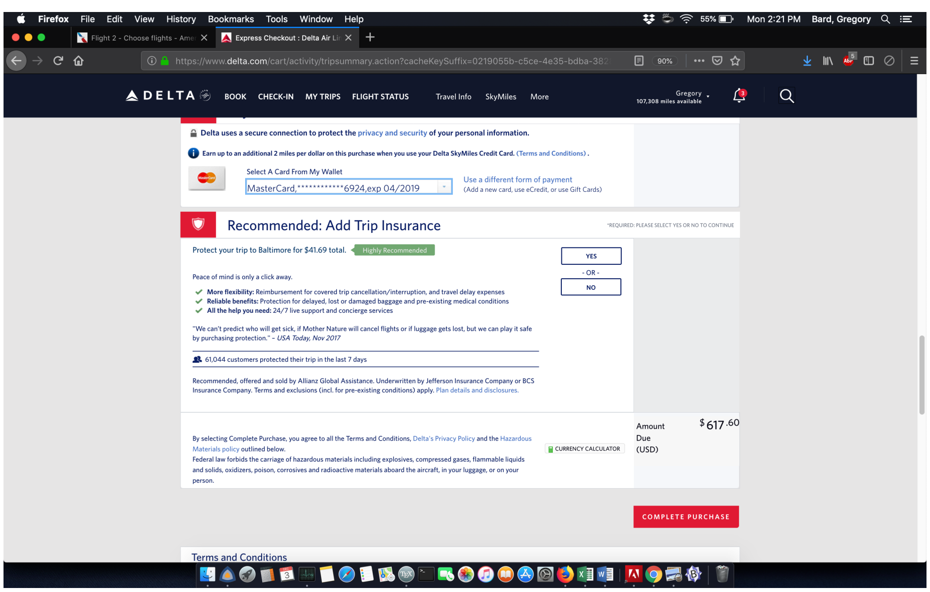
Next, I should find out when this expected value will be positive. The expected value is positive when

$$\begin{aligned} 617.6p - 41.69 &> 0 \\ 617.6p &> 41.69 \\ p &> 41.69/617.6 \\ p &> 0.0675032\cdots \end{aligned}$$

This insurance policy is attractive if and only if the probability that I am too sick to fly, (or that I suffer a similar hazard covered by the insurance policy, such as a loved one becoming sick), exceeds 6.75032%, because that's the range of probabilities that will make the expected value positive. (Recall that in industry, probabilities are typically reported to the nearest basis point, so we would write this as 6.75% in the workplace.)

Honestly, that sounds too high. I would not suffer catastrophic inconvenience if I had to pay \$ 617.60, and I am confident that the expected value is negative, because I am confident that $p < 6.75\%$.

If you're curious, this is what the before-mentioned offer looks like.



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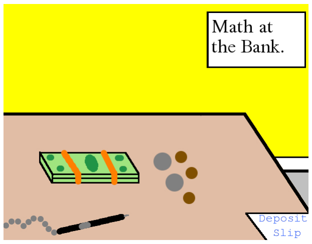


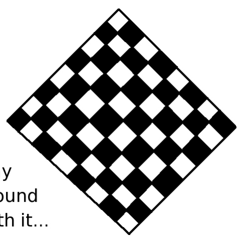
When you rent a car in the USA, especially at an airport, the car-rental company will frequently try to sell you extra insurance—sometimes with great eagerness. Some standard car-insurance policies cover a driver not only when driving their own car, but also when driving a rental car. Other standard car-insurance policies do not.

The following two problems will model some decision making that I made myself when traveling to New Jersey, to visit my mother. As it turns out, the car-insurance policy that I have for my own car covers me whenever I rent a car.

When I got to the rental car desk, the agent offered me the extra insurance (as always). So, I explained that I was already covered and that I wanted to decline the extra insurance. Nonetheless, the agent persisted.

He pointed out that if I were to get into a car accident, then they would have to collect the deductible of my insurance policy from me. This is a good sales tactic, because for a lot of people, having to suddenly pay \$ 500, \$ 1000, or \$ 2000, when one was not expecting to do so, is extremely inconvenient. It might or might not rise to the level of catastrophic inconvenience, depending on personal finances.

As for myself, at this point in my life, I am grateful that a sudden unplanned expenditure of \$ 1000 would be only slightly inconvenient, not catastrophically inconvenient. That means I should compute the expected value, in order to decide if I should accept or decline.



Play
Around
With it...

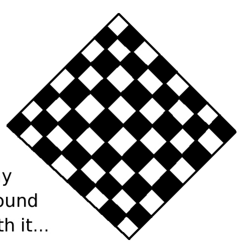
3-4-12

The first case of my mental computations regarding the extra coverage in the previous box occurred in May. While I normally live in rural Wisconsin, I had landed in Newark, New Jersey, and I was going to be driving in the industrial suburbs of New Jersey near New York City. (For readers who are unfamiliar, let me summarize by saying that these two locations represent the opposite extremes of the spectrum from polite drivers to aggressive drivers. Many comedians joke about how bad drivers in New Jersey are.)

My deductible at that time was \$ 1000. The rental car company wanted \$ 17 for one day of coverage.

What probability would cause this insurance to have a positive expected value?

The answer will be given on Page 321 of this module.



Play
Around
With it...

3-4-13

On the next visit a few months later, I felt more confident. For unrelated reasons, I had changed my policy to have a \$ 500 deductible. The charge was still \$ 17 per day, but my trip was for three days.

What would the probability need to be, for me getting into a car accident during the trip, for this insurance to have positive expected value? I should emphasize that I'm not thinking of this as a probability per day, but as a probability over the whole trip.

The answer will be given on Page 321 of this module.

Citation: The following problem is a quotation from *Finite Mathematics & Its Applications*, 10th edition, by Larry Goldstein, David Schneider, and Martha Siegel, where it was Exercise 7-4-35. This problem was marked "PE" which means it was quoted from some sort of professional examination, such as might be taken to become an actuary.

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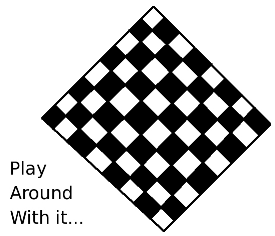


3-4-14

The promoter of a football game is concerned that it will rain. She has the option of spending \$ 8000 on insurance that will pay \$ 40,000 if it rains. She estimates that the revenue from the game will be \$ 60,000 if it does not rain and \$ 25,000 if it does rain.
What must the chance of rain be if she is ambivalent about this insurance?

(a) 20% (b) 25% (c) 30% (d) 35% (e) 40%

For the sake of practice, use the method of four rows to make one table that describes what happens if the insurance is not purchased, and another table that describes what happens if the insurance is purchased. After that, find the expected value, which is a function of p, in each case.

The answers will be presented over the next few boxes. Then I will show you how to find the probability.

Don't read the solution before making an honest attempt at solving the problem in the previous box.

Continuing with the previous checkerboard box, here is the table if she doesn't get the insurance:

Outcome	It Rains	It Doesn't Rain
Probability	p	$1-p$
Payoff	$25,000$	$60,000$
Product	$25,000p$	$60,000(1-p)$

Therefore, the expected value is

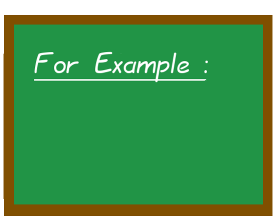
$$25,000p + 60,000(1-p) = 25,000p + 60,000 - 60,000p = 60,000 - 35,000p$$

Continuing with the previous boxes, here is the table if she does get the insurance:

Outcome	It Rains	It Doesn't Rain
Probability	p	$1-p$
Payoff	$25,000+40,000-8000 = 57,000$	$60,000-8000 = 52,000$
Product	$57,000p$	$52,000(1-p)$

Therefore, the expected value is

$$57,000p + 52,000(1-p) = 57,000p + 52,000 - 52,000p = 52,000 + 5000p$$



3-4-15

Over the last several checkerboard boxes, we obtained an expected value of $60,000 - 35,000p$ if she doesn't get the insurance, and $52,000 - 5000p$ if she does get the insurance.

Now we will find the value of p which will make her ambivalent.

$$\begin{aligned} 52,000 + 5000p &= 60,000 - 35,000p \\ 52,000 + 40,000p &= 60,000 \\ 40,000p &= 8000 \\ p &= 8000/40,000 \\ p &= 1/5 \end{aligned}$$

Therefore, we select (a) 20%.

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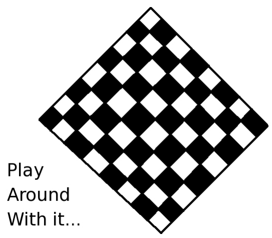
Now you know that you're studying real-world phenomena. You're solving problems quoted directly from tests that are taken for professional certifications!

Citation: The following problem is a quotation from *Finite Mathematics & Its Applications*, 10th edition, by Larry Goldstein, David Schneider, and Martha Siegel, where it was Exercise 7-4-34. This problem was marked "PE" which means it was quoted from some sort of professional examination, such as might be taken to become an actuary.



3-4-16

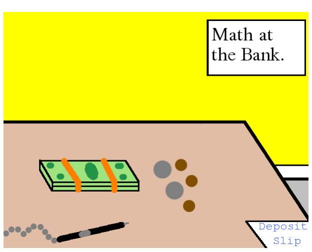
Bob wishes to insure a priceless family heirloom against theft. The annual premium for Policy A is \$ 150, and it will pay \$ 75,000 if the heirloom is stolen. Policy B will pay \$ 100,000, but the annual premium for Policy B is \$ 250. Bob estimates the probability that the heirloom will be stolen in any given year and remains undecided between the two policies. What is the estimated probability?

(a) 0.001 (b) 0.002 (c) 0.003 (d) 0.004 (e) 0.005

Again, for the sake of practice, I want you to make three tables.

- First, if Bob does not obtain insurance at all.
- Second, if Bob chooses Policy A.
- Third, if Bob chooses Policy B.
- Compute the expected value in all three cases.
- Fourth, find the probability that makes Bob ambivalent between Policy A and Policy B.
- Fifth, go back and turn the expected values into dollar values.

Also, do you know why it was important that the word "priceless" be included in the wording of the problem? I'll answer that question first, and then present the solutions to all the above sub-problems.



The reason it was important to include the word priceless is that Bob has to prove that the work of art is worth more than \$ 100,000 plus his deductible to claim all \$ 100,000 from Policy B. If the work of art is worth less than that, then they'll pay him what it is worth minus the deductible. Similarly, Policy A will pay \$ 75,000 if the work of art is provably worth more than \$ 75,000 plus his deductible. Otherwise, they'll pay him what it is worth, minus the deductible.

The word priceless here is the exam telling the potential actuary that it is safe to assume that the true value of the artwork is well above \$ 100,000, perhaps something like \$ 200,000 or \$ 300,000. It is not meant to convey the idea that the artwork is priceless in the sense of a historical artifact that is the sole surviving example of its kind, like the Cyrus Cylinder from 534 BCE, or the Archimedes Palimpsest.

Finally, since no one would hang a \$ 200,000 piece of artwork in a house worth less than a few million dollars, we know that this theft would not be a catastrophic loss for Bob.

(If you don't know the story of the Archimedes Palimpsest, look it up on *Wikipedia*—it is an amazing story, but totally unrelated to discrete mathematics, so I will not describe it here.)

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If Bob does not get insurance, then the table looks like this:

Outcome	Stolen	Not Stolen
Probability	p	$1-p$
Payoff	0	0
Product	0	0

The expected value is clearly zero. (Of course, on a timed test, you should not waste time writing out such a table unless you are instructed to do so.)

If Bob gets Policy A, then the table looks like this:

Outcome	Stolen	Not Stolen
Probability	p	$1-p$
Payoff	$75,000-150$	-150
Product	$74,850p$	$-150(1-p)$

The expected value is

$$74,850p - 150(1-p) = 74,850p - 150 + 150p = 75,000p - 150$$

If Bob gets Policy B, then the table looks like this:

Outcome	Stolen	Not Stolen
Probability	p	$1-p$
Payoff	$100,000-250$	-250
Product	$99,750p$	$-250(1-p)$

The expected value is

$$99,750p - 250(1-p) = 99,750p - 250 + 250p = 100,000p - 250$$

As you've probably noticed by now, we can save a lot of time by jumping to the shortcut formula:

$$\text{expected value} = (\text{pay off})(p) - (\text{premium})$$

Since the actuarial exams are timed, with great time pressure, this is exactly the type of shortcut that separates the student who passes on the second try from the student who needs five or more tries.

Continuing with the long problem about Bob's artwork, since Bob is ambivalent between the two policies, we set them equal to each other and solve for p.

$$\begin{aligned} 75,000p - 150 &= 100,000p - 250 \\ -150 &= 25,000p - 250 \\ 100 &= 25,000p \\ 100/25,000 &= p \\ 0.004 &= p \end{aligned}$$

Therefore, we select (d) 0.004 for the probability. We're almost done, and we'll continue in the next box.

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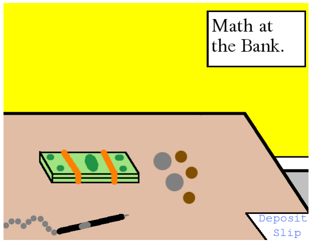


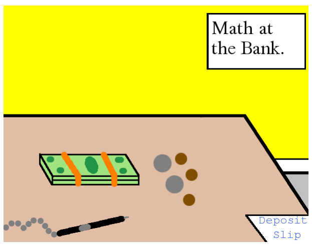
The expected value of Policy A becomes

$$75,000(0.004) - 150 = 300 - 150 = 150$$

and the expected value of Policy B becomes

$$100,000(0.004) - 250 = 400 - 250 = 150$$

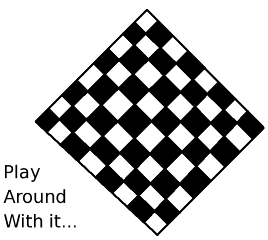
an exact match, confirming our work. Meanwhile, the expected value of not getting insurance is 0. That explains why Bob is not considering the possibility of going uninsured, because both insurance policies have a positive expected value.



I'd like to give you a taste of stock options. In the next several boxes, we will explore an imaginary company, called General Mechanics. Suppose common stock in General Mechanics is currently trading at $ 14.50 per share. There is a forecast from a financial advisory firm about the price that General Mechanics will be trading at a year from now.

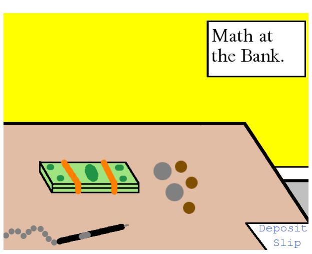
Price (dollars per share)	13	14	15	16	17	18	19
Probability	1/16	2/16	3/16	4/16	3/16	2/16	1/16

As you can see, I'm using round numbers to keep things simple. We'll continue in the next box.



3-4-17

As a warm up, compute the expected value of a share of General Mechanics common stock a year from now, according to the financial advisory firm's data, which is given in the previous box. [Answer: $ 16.00 per share.]



Is the above problem realistic? Aren't you getting "something for nothing" by buying the stock? After all, it is trading for $ 14.50 per share now, and according to the financial advisory firm, it will trade for $ 16 per share a year from now.

To see if this is realistic, let's compute the yield. We can easily calculate

$$16/14.50 = 1.10344\cdots = 1 + 0.103448\cdots$$

, so that's a yield of 10.34%. The normal rates used for planning retirement or investing are that the S&P 500 returns 11% nominal rate of return, and 7% real rate of return (purchasing power parity). Don't worry if you don't know what the real rate of return means, or what purchasing power parity means. The 11% rate of return is like an interest rate that you would use when computing simple interest or compound interest. So there's nothing unrealistic about the 10.34%, especially since considerable risk is involved.

However, stock prices are not integers. When I was growing up, stock prices were traded in intervals of an eighth of a dollar. So a stock could be $13, 13\frac{1}{8}, 13\frac{1}{4}, 13\frac{3}{8}, 13\frac{1}{2}13\frac{5}{8}$, *et cetera*, but nothing in between. At some point, they allowed sixteenths of a dollar. Then on April 9th, 2001, the Securities & Exchange Commission ordered that the prices be given in units of one cent (instead of 1/8th of a dollar or 1/16th of a dollar). I used integers to keep the problem simple.

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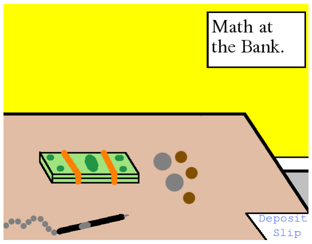
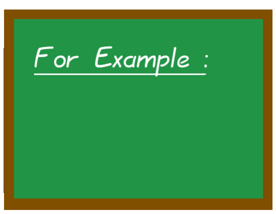


A *call option* on a stock gives me the right (but not the obligation) to purchase this particular stock, at any point until the option's expiration date, for a fixed and specified price called "the strike price." Now suppose someone wants to sell me a call option, at a price of \$ 1.17 per share, that expires one year from now, with a strike price of \$ 15 per share.

The stock is currently trading at \$ 14.50 per share, so even if I choose to buy the option, I'm definitely not going to exercise it right away. That would be crazy, because the stock is trading for \$ 14.50 per share, which means I can buy it on the market at \$ 14.50 per share instead of \$ 15 per share.

However, a year from now, it might be the case that the stock has risen in value to \$ 17 or \$ 18. In that case, I'd be happy to buy the shares for \$ 15 per share.

We'll analyze this precisely, in the next box.



3-4-18

Let's construct our 4-row table, for analyzing the expected value of the stock option described in the previous box. The price now serves in the role of the name of the outcome. The probabilities are unchanged. The profit is a bit tricky to figure out.

- The strike price is \$ 15, so if the stock is trading for \$ 13 or \$ 14, then it would be crazy to exercise the option. Why pay \$ 15 when I can pay \$ 13 or \$ 14 by purchasing the stock on the open market?! In these two cases the option is worthless.
- The option is also worthless if the stock is trading for \$ 15, because the option is not giving me any capability that I didn't have already.
- If the stock is trading for \$ 19 per share, then I could exercise the option, buying the stock at \$ 15 per share, and immediately sell it at the market rate of \$ 19 per share. This gives me a profit of \$ 4 per share.
- Similarly, if the stock is trading for \$ 18 per share, I could exercise the option, and immediately sell the stocks, for a profit of \$ 3 per share.
- As you might guess, if the stock is trading for \$ 17 per share, the profit is worth \$ 2 per share. Finally, if the stock is trading for \$ 16 per share, the profit is worth \$ 1 per share.

The table of four rows is given in the next box.

Continuing with the previous box, we now have the following table:

Stock Price (dollars per share)	13	14	15	16	17	18	19
Probability	1/16	2/16	3/16	4/16	3/16	2/16	1/16
Profit Obtained (dollars per share)	0	0	0	1	2	3	4
Product (dollars per share)	0	0	0	4/16	6/16	6/16	4/16

The expected value is

$$\frac{4}{16} + \frac{6}{16} + \frac{6}{16} + \frac{4}{16} = \frac{20}{16} = \$\ 1.25/\text{ share}$$

If my experts are correct, the fair market value of this call option is \$ 1.25. Since it is available for only \$ 1.17, I should buy this option—assuming that I trust the financial advisory firm.

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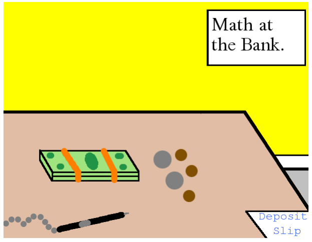
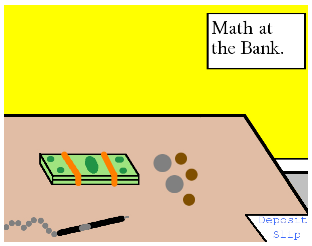


As was stated on Page 319, a *call option* on a stock gives me the right (but not the obligation) to purchase this particular stock, at any point until the option's expiration date, for a fixed and specified price called "the strike price." You can think of a call option like betting that a stock will go up in price.

If you strongly believe that a stock will go up in price, you should buy shares of the stock. However, if you have intermediate confidence, then you can buy a call option instead of buying shares. If the stock goes up, you make money. If the stock doesn't go up, you have only lost the cost of the option. The amount of capital that you are risking is a lot less.

Imagine two twins, Ned and Ted. Suppose Ned buys 100 shares (costing him \$ 1500), and Ted buys a call option for 100 shares (costing him \$ 117). Further suppose that, due to unforeseen circumstances, the stock falls all the way to \$ 3 per share. Ned's stocks are now worth only \$ 300, so Ned has lost \$ 1200. On the other hand, Ted has lost nothing except the \$ 117 he paid for the option.



The analysis presented above is oversimplified. In reality, the stock price can take on many possible values, one cent apart. There are other considerations as well, such as the variance of the portfolio, which is a fairly good measure of non-catastrophic risk.

Therefore, I will not give you more problems to do about stock options, because we don't have the time or space to develop the true theory, and it seems like a waste of time to practice a primitive theory.

This module is now complete. Thank you for reading. All that follows are the solutions to various problems given earlier in the module.

Here is the table for the question (from Page Page 302) about females in the census report, (the one which asked respondents about how many times they had been married).

Outcome	Never Married	Married Once	Married Twice	Married 3 or more times
Probability	0.246	0.587	0.136	0.031
Marriages	0	1	2	3
Product	0	0.587	0.272	0.093

The expected value is $0 + 0.587 + 0.136 + 0.031 = 0.952$.

Here is the answer to the problem from Page 305 of this module, about shipping books from New York to Greece.

By the way, the names of the outcomes are arbitrary for this problem.

Outcome	Small	Medium	Large	Extra-Large
Probability	1134/2426	847/2426	328/2426	117/2426
Mass/Weight	0.6	1.5	2.1	2.8
Product	$0.280461\cdots$	$0.523701\cdots$	$0.283924\cdots$	$0.135037\cdots$

The expected value is the sum of the fourth row, and is $1.22312\cdots$ pounds. Typically, since the weights of each size were reported to two significant digits, then the expected value would be reported to three significant digits, namely as 1.22 pounds. However, in a homework problem, it is the policy of this textbook to use six significant digits. I should also mention that it is okay to present the columns in any order that might seem suitable.

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Here are the solutions to the problem from Page 310, where I asked you to compute $E[X^2]$ and $E[X]^2$ of a Bernoulli random variable.

- What is $E[X^2]$? [Answer: p.]
- What is $E[X]^2$? [Answer: p^2.]

As a consequence, this means that

$$Var[X] = p - p^2 = p(1-p) = pq$$

and the standard deviation is therefore $\sqrt{p(1-p)} = \sqrt{pq}$. That's why $\sqrt{p(1-p)} = \sqrt{pq}$ shows up in so many different places when discussing Bernoulli random variables, such as the Bernoulli-DeMoivre-Laplace inequalities.

Here is the solution to first extra-car-rental-insurance problem from Page 314.

Since $17/1000 = 0.017$, if $p > 0.017$ then the extra insurance has a positive expected value. That's what I computed mentally, since dividing by 1000 in one's head is easy.

It also helps that it was raining. In the rain, driving in industrial New Jersey after not having done so in several years, is the probability of my getting into an accident greater than 1.7%? I thought it was, so I said yes to the insurance.

We don't need the method of four rows for this problem, but sometimes it is nice to see what that method would produce.

We could still use the method of four rows, if we wanted to be formal.

Outcome	Accident	No Accident
Probability	p	$(1-p)$
Payoff	$1000 - 17 = 983$	-17
Product	$983p$	$-17(1-p)$

The expected value is

$$983p - 17(1-p) = 983p - 17 + 17p = 1000p - 17$$

Therefore, if $1000p - 17 > 0$, or equivalently if $1000p > 17$, the expected value is positive. This means that if $p > 0.017$, then the extra insurance has a positive expected value.

Here is the solution to the second extra-car-rental-insurance problem, which was on Page 314.

Notice that the premium is now $(3)(17) = 51$. Since $51/500 = 0.102$, if $p > 0.102$ then the extra insurance has a positive expected value. The easy mental trick for dividing by $ 500 is to double and then shift the decimal point by three spots. The double of 51 is 102 before shifting, and 0.102 after shifting.

Is the probability of my getting into a car accident at any point during the three days greater than 10.2%?! Definitely not. It also helps that the weather was clear, and that I felt more confident on this trip. Therefore, I declined the extra insurance.

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