Micro-Module 4.4: The Missing Principle of Combinatorics

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1 Two Fundamental Questions and an Incomplete Chart

Over the last three modules, we've seen that the two fundamental questions in a combinatorics question are "does order matter?" and "are repeats allowed?" Then, using the answers from those two questions, we can choose which principle we should be using. For review, here is that table.

	Repeats Allowed	Repeats not Allowed
Order Does Matter	Exponent Princ	Permutation Princ
Order Doesn't Matter	?!?	Combinations Princ

Over all, the table above can be used to solve almost all combinatorial problems. Yet it has four places, and we haven't talked about "the fourth case" yet. (The position which it occupies is marked by "?!?" in the table.) While the "fourth case" is rare, we will talk about that in this document.

A Humorous Note:

While I have your attention, it is good to remind you of the factorial principle: "There are $n!$ possible orderings of n objects." A funny thing about the factorial principle is that while it rarely comes up in problem solving, when instructors ask students months or years after this course about which technique should be applied in a specific situation, students guess the factorial principle really often. That's like being asked to guess which US State a stranger is from, and instead of guessing populous states first (New York, California, Texas, ...), someone decides to start with Wyoming, the least populous state. (Presumably the fact that the stranger is from a US State is already known from earlier in the conversation.)

2 Stating the Fourth Case

The formula for the fourth case isn't too hard. Some students choose to memorize it and just dispense with the derivation. Memorizing a formula isn't a terrible strategy, but students usually do better when they learn where a formula comes from. When assigning n objects (or people) into k categories, where order doesn't matter, and categories can be repeated, there are

$$C_{n+k-1,k-1} = C_{n+k-1,n}$$

possible choices. It should be noted that "k categories" is the distinguishing feature of this principle, among the several principles of combinatorics that we've seen so far. (There was nothing equivalent to "categories" for those principles.)

Usually, it is a good idea to study the derivation of a formula. However, my derivation is one that few students like and most students find unsatisfying. Therefore, I recommend you view the derivation of Prof. Oscar Levin (see below) instead. However, if you want to see my derivation, it is given in Section 3.2 of this module.

3 Two Good Derivations

3.1 Prof. Oscar Levin's Derivation of the Fourth Case

My friend Prof. Oscar Levin of the University of Northern Colorado has written a very nice textbook, *Discrete Mathematics—An Open Introduction.* His book is free in electronic form, and it has won the endorsement of the Open Textbook Initiative of the American Institute of Mathematics, the same award that my second commercially published textbook, *Sage for Undergraduates*, published by the American Mathematical Society in 2015, has won. To see his derivation, follow these instructions:

- Go to this URL: `http://discrete.openmathbooks.org/dmoi3/dmoi.html`
- Click on Chapter 1: "Counting."
- Click on Section 1.5: "Stars and Bars," and read it. (That section is roughly 2.25 pages long.)

3.2 Prof. Gregory Bard's Derivation of the Fourth Case

Imagine a freshman class at a high school, with 1000 freshmen, and they have their choices of studying Spanish, French, German, or Latin. The principal isn't worried, just yet, at figuring out which students are taking which languages. At this moment, the principal just wants to know about the total enrollments in each language, to help with setting the schedule. How many possible enrollment structures are there?

Imagine that we have 3 dividers, and 1000 chairs, that we will position into 1003 slots. These will allow us to figure out how many students are in each language. For example, let `h` be a chair, and let `|` be a divider. We'll seat the students in the order Spanish, French, German, and Latin, with a divider between each language group.

Consider

$$\underbrace{\text{hh}\cdots h}_{995}||\text{hhh}|\text{hh}$$

which means 995 take Spanish, 0 take French, 3 take German, and 2 take Latin. Similarly

$$\underbrace{\text{hh}\cdots h}_{993}|\text{h}|\text{hhh}|\text{hhh}$$

means 993 take Spanish, 1 takes French, 3 take German, and 3 take Latin. Finally

$$\underbrace{\text{hh}\cdots h}_{991}|\text{hh}||\text{hhhhhhh}$$

means 991 take Spanish, 2 take French, 0 take German, and 7 take Latin.

As you can see, each sequence of chairs and dividers makes a enrollment structure, and each possible enrollment structure makes a sequence of chairs and dividers. With 1000 chairs and 3 dividers, we have 1003 objects.

Interestingly, if you tell me where the 3 dividers go, I automatically know where the 1000 chairs go, and there are $C_{1003,3}$ ways to place the dividers.

Similarly, if you tell me where the 1000 chairs go, I automatically know where the 3 dividers go, and there are $C_{1003,1000}$ ways to place the chairs.

Therefore, we know that there are

$$C_{1003,3} = C_{1003,1000}$$

possible enrollment structures.

We get the general formula by replacing 1000 with n, 3 with $k-1$ and 1003 with $n+k-1$. The general formula, therefore, is

$$C_{n+k-1,k-1} = C_{n+k-1,n}$$

4 Some Examples to Think About

1. Suppose that each night on a cruise ship, the dinner special can be beef, chicken, or fish.
   - For a seven day cruise, how many possible sequences of dinner specials can there be, if we care about the ordering? In other words, BCCFBC is considered different from BBCCCF.
   - How about if we don't care about the ordering, and just care about how many beef days, how many chicken days, and how many fish days there are?
2. Suppose a university has 50 math majors, 250 computer-science majors, 80 cybersecurity majors, and 150 computer-engineering majors.
   - Treating all students as individuals, ignoring their majors, in how many ways can a panel of 12 students be assembled? (In other words, a panel is a list of names in this sub-problem.)
   - Treating any two students with the same major as entirely interchangeable, in how many ways can a panel of 12 students be assembled? (For example, 1 math major, 3 computer-science majors, 2 cybersecurity majors, and 6 computer-engineering majors is one possible panel.)
3. A bin of balls[1] has an ample supply of green balls, blue balls, purple balls, and yellow balls. I'm going to grab a dozen, put them in a paper bag, take them home, and dump them out. How many possible outcomes are there, for what comes out of the bag?
4. How many solutions to $x_1 + x_2 + x_3 + x_4 = 12$ are there, if we restrict the x_is to being non-negative integers?
5. Why should one say that three questions above this question are actually asking the same thing?

Note: Be certain to read what Prof. Oscar Levin has written about this subject, using the directions provided in Section 3.1 above. He has some nice problems for you there.

5 Solutions to the Examples

1. Suppose that each night on a cruise ship, the dinner special can be beef, chicken, or fish.
   - For a seven day cruise, how many possible sequences of dinner specials can there be, if we care about the ordering? In other words, BCCFBC is considered different from BBCCCF.

     $$3^7 = 2187$$

   - How about if we don't care about the ordering, and just care about how many beef days, how many chicken days, and how many fish days there are?

     $$C_{7+3-1,3-1} = C_{9,2} = 36 = C_{9,7} = C_{7+3-1,7}$$

2. Suppose a university has 50 math majors, 250 computer-science majors, 80 cybersecurity majors, and 150 computer-engineering majors.
   - Treating all students as individuals, ignoring their majors, in how many ways can a panel of 12 students be assembled? (In other words, a panel is a list of names in this sub-problem.)

     $$C_{530,12} = 904,682,243,786,771,522,692,100$$

     but note that $C_{530,12}$ is a more useful name than $904,682,243,786,771,522,692,100$ for that number, though both names are valid.

[1] Tradition would have me draw the balls from an urn rather than from a bin, but since the mid-19th century, the word urn is only used for the container of a cremated person's ashes, and therefore I decline to use the word urn. The `xkcd.com` cartoonist Randall Monroe has something to say on that point as well. See the last page of this module.

- Treating any two students with the same major as entirely interchangeable, in how many ways can a panel of 12 students be assembled? (For example, 1 math major, 3 computer-science majors, 2 cybersecurity majors, and 6 computer-engineering majors is one possible panel.)

$$C_{12+4-1,4-1} = C_{15,3} = 455 = C_{15,12} = C_{12+4-1,12}$$

3. A bin of balls has an ample supply of green balls, blue balls, purple balls, and yellow balls. I'm going to grab a dozen, put them in a paper bag, take them home, and dump them out. How many possible outcomes are there, for what comes out of the bag?

$$C_{12+4-1,4-1} = C_{15,3} = 455 = C_{15,12} = C_{12+4-1,12}$$

   Note that putting the balls in a paper bag means that any information about the order is destroyed, because the balls roll around inside the bag. That's how you know that order doesn't matter in this problem.

4. How many solutions to $x_1 + x_2 + x_3 + x_4 = 12$ are there, if we restrict the x_is to being non-negative integers?

$$C_{12+4-1,4-1} = C_{15,3} = 455 = C_{15,12} = C_{12+4-1,12}$$

5. Why should one say that the three questions above this question are actually asking the same thing? Suppose that math majors are represented by yellow balls, cybersecurity majors by green balls, computer-scientists are blue balls, and computer-engineers are purple balls. The sack of twelve balls represents the panel of twelve students. Not caring who the students are, but only about their major is analogous to caring about the colors of the balls, but not caring about which balls I've brought home with me. Similarly, the value of x_1 could represent yellow balls, the value of x_2 could represent green balls, the value of x_3 could represent blue balls, and the value of x_4 could represent purple balls.

Here is a comic from the `xkcd.com` cartoonist Randall Monroe. I'm sharing it here via The Creative Commons, and this should be considered an academic citation.